AVERAGING ALONG DEGENERATE FLOWS ON THE ANNULUS

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Abstract. Periodic flows on the annulus are considered. For flows that degenerate (i.e. the flow becomes arbitrarily slow along some flow lines) a convergence rate for the averaging of functions along the flow is obtained. This rate – which is slower than the rate obtained for flows that do not degenerate (i.e. when there is a spectral gap) – holds on an appropriate functional subspace. The main ingredient is an estimate of the density of the spectrum of the generator near zero.

MSC (2010): 37A30 (primary); 35P20, 35B40

Keywords: degenerate flows; density of states

Acknowledgements: The authors acknowledge support from Fellowship EP/N020154/1 of the Engineering and Physical Sciences Research Council (EPSRC).

Date: July 12, 2019.

Contents

1. Introduction 1
2. Spectral analysis and fibered operators 3
3. Toy example 5
4. Flow in an annulus: Proof of Theorem 1.1 7
References 15

1. Introduction

In this paper we study self-adjoint operators of the form

$$A = -i\varphi(m)\frac{\partial}{\partial\theta} \qquad \text{acting in } L^2(\mathcal{A}) \tag{1.1}$$

where $\mathcal{A} = [0,1]_m \times \mathbb{S}^1_\theta$ is an annulus and where $\varphi \geq 0$ is continuous. The operator A is the generator of a periodic flow in $\mathcal{A}$ and φ measures the speed of the flow along the m-fiber. We are interested in obtaining a rate at which the time averages

$$P^T f := \frac{1}{2T}\int_{-T}^{T} e^{itA} f \,\mathrm{d}t = \frac{1}{2T}\int_{-T}^{T} f(m, t\varphi(m))\,\mathrm{d}t$$

converge to the spatial average

$$Pf := \int_{\mathbb{S}^1} f(m,\theta)\,\mathrm{d}\theta \tag{1.2}$$

as $T \to +\infty$ in the degenerate case where $\min_{m\in[0,1]} \varphi(m) = 0$. In this case there cannot be a uniform rate on $L^2(\mathcal{A})$ and our main task is to identify a subspace $\mathcal{X} \subset L^2(\mathcal{A})$ on which a uniform rate *does* hold.

1.1. **Main result.** Our main result is:

Theorem 1.1. *Assume that there exist $m_0, c, \alpha > 0$ such that $\varphi(m) = cm^\alpha$ on $[0, m_0)$ and that φ is continuous and bounded uniformly away from 0 on $[m_0, 1]$. Then there exists a subspace $\dot{\mathcal{H}}_0^{s,\gamma} \subset L^2(\mathcal{A})$ (defined below in (4.3)[1]) and a constant $C > 0$ such that the following uniform rate holds:*

$$\|P^T - P\|_{\dot{\mathcal{H}}_0^{s,\gamma}\to L^2(\mathcal{A})} \le CT^{-\frac{s}{s+\alpha}}, \qquad \forall T > 1,$$

where $s > 1/2$ and $\gamma \ge 0$ satisfy $\gamma + s/\alpha > 1/2$, and where C does not depend on T.

Strategy of the proof. The strong convergence (i.e without a rate) $P^T \to P$ follows immediately from von Neumann's ergodic theorem [vN32]. Von Neumann's idea was to use the spectral theorem to write $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda)$, where $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is the resolution of the identity of A. This leads to

$$\begin{aligned}(P^T - P)f &= \frac{1}{2T}\int_{-T}^{T} e^{itA} f \, \mathrm{d}t - Pf = \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}} e^{it\lambda} \, \mathrm{d}E(\lambda)f \, \mathrm{d}t - Pf \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}\setminus\{0\}} e^{it\lambda} \, \mathrm{d}E(\lambda)f \, \mathrm{d}t = \int_{\mathbb{R}\setminus\{0\}} \frac{\sin T\lambda}{T\lambda} \, \mathrm{d}E(\lambda)f. \end{aligned} \tag{1.3}$$

This last expression tends to 0 as $T \to +\infty$. It is well-known that a spectral gap leads to the rate T^{-1}. However here, owing to the assumption that the flow degenerates, there is no spectral gap. In [BAM19] we showed that a rate may be extracted if there exists a subspace $\mathcal{X} \subset L^2(\mathcal{A})$ and some $r > 0$ such that the density of states (DoS) has a bound of the form

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(E(\lambda)f, g\right)_{L^2(\mathcal{A})} \right| \le \psi(\lambda)\|f\|_{\mathcal{X}}\|g\|_{\mathcal{X}}, \qquad \forall f, g \in \mathcal{X}, \, \forall\lambda \in (-r, r) \setminus \{0\}, \tag{1.4}$$

where $\psi \in L^1(-r, r)$ is strictly positive a.e. on $(-r, r)$. This then allows one to replace the integration $\mathrm{d}E(\lambda)$ in (1.3) with $\frac{\mathrm{d}E(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda$ and to apply the estimate (1.4) near $\lambda = 0$. Therefore, to prove Theorem 1.1 the main task is to obtain an estimate of the form (1.4), which involves identifying an appropriate subspace $\mathcal{X}$. This is achieved thanks to the observation that the operator A is unitarily equivalent to the multiplication operator φk via a Fourier transform in θ (where $k \in \mathbb{Z}$ is the Fourier conjugate of θ).

To obtain an estimate of the DoS as in (1.4), the first step is to understand the structure of the spectrum. Here we take the point of view that A is fibered in m, composed of the

[1]Roughly speaking, $\dot{\mathcal{H}}_0^{s,\gamma}$ consists of functions that have γ derivatives in θ, whose $\dot{H}^\gamma$ norm is $(s - \frac{1}{2})$–Hölder continuous in m and are constant along the fiber $m = 0$.

one-dimensional operators

$$A(m) = -i\varphi(m)\frac{\mathrm{d}}{\mathrm{d}\theta} \qquad \forall m \in [0,1], \tag{1.5}$$

acting on the circle $\mathbb{S}^1$. This point of view is common in the context of the Euler equations, see [Cox14] for instance. The main difficulty is now evident: the spectrum of each fiber $A(m)$ is discrete, while the spectrum of A may have discrete, absolutely continuous and singular continuous parts.

1.2. **Organization of the paper.** In Section 2 we discuss in detail various properties of *fibered* self-adjoint operators. In Section 3 we present a toy model which mimics some of the behavior of the operator A in order to gain a better intuition. Then, in Section 4 we turn our attention to the flow in an annulus, first proving a bound on the DoS (Theorem 4.7) and then obtaining a rate for the associated ergodic theorem, proving Theorem 1.1.

2. Spectral analysis and fibered operators

In this section we recall some properties of self-adjoint operators and of self-adjoint fibered operators, and prove some results which are not readily available in the standard literature. Our discussion remains as general as possible, working with abstract self-adjoint operators in abstract Hilbert spaces. Our flow operator defined above is a special example and does not appear in this section.

2.1. **Resolution of the identity.** Since the spectral theorem plays an essential role in our proof, it is worthwhile recalling the definition of the resolution of the identity of a self-adjoint operator.

Let $\mathcal{H}$ be some Hilbert space, and let $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Its associated spectral family $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is a family of projection operators in $\mathcal{H}$ with the property that, for each $\lambda \in \mathbb{R}$, the subspace $\mathcal{H}^\lambda = E(\lambda)\mathcal{H}$ is the largest closed subspace such that

(1) $\mathcal{H}^\lambda$ *reduces* H, namely, $HE(\lambda)g = E(\lambda)Hg$ for every $g \in D(H)$. In particular, if $g \in D(H)$ then also $E(\lambda)g \in D(H)$.
(2) $(Hu, u)_{\mathcal{H}} \le \lambda(u, u)_{\mathcal{H}}$ for every $u \in \mathcal{H}^\lambda \cap D(H)$.

Given any $f, g \in \mathcal{H}$ the spectral family defines a complex function of bounded variation on the real line, given by

$$\mathbb{R} \ni \lambda \mapsto (E(\lambda)f, g)_{\mathcal{H}} \in \mathbb{C}.$$

It is well-known that such a function gives rise to a complex measure (depending on f, g) called the *spectral measure*. Recall the following useful fact:

Proposition 2.1 ([Kat95, X-§1.2, Theorem 1.5]). *Let $U \subset \mathbb{R}$ be open. The set of $f, g \in \mathcal{H}$ for which the spectral measure is absolutely continuous in U with respect to the Lebesgue measure forms a closed subspace $\mathcal{AC}_U \subset \mathcal{H}$. This subspace is referred to as the* absolutely continuous subspace of H on U.

2.2. **The density of states.** Let $\mathcal{AC}_U \subset \mathcal{H}$ be the absolutely continuous subspace of H on U and let $\lambda \in U$. If there exists a subspace $\mathcal{X} \subset \mathcal{AC}_U$ equipped with a stronger norm such that the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)\cdot,\cdot)_{\mathcal{H}} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is bounded then it induces a bounded operator $B(\lambda) : \mathcal{X} \to \mathcal{X}^*$ defined via

$$\langle B(\lambda)f, g\rangle_{(\mathcal{X}^*,\mathcal{X})} = \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f, g)_{\mathcal{H}}, \quad f, g \in \mathcal{X},$$

where $\langle\cdot,\cdot\rangle_{(\mathcal{X}^*,\mathcal{X})}$ is the $(\mathcal{X}^*, \mathcal{X})$ dual-space pairing, and $\mathcal{X}^*$ is the dual of $\mathcal{X}$ with respect to the inner-product on $\mathcal{H}$.

Definition 2.2. *We refer to both the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)\cdot,\cdot)_{\mathcal{H}}$ and the operator $B(\lambda)$ as the* density of states (DoS) *of the operator H at λ.*

In physics, the DoS at λ represents the number possible states a system can attain at the energy level λ.

2.3. **Fibered operators.** We follow the notation of [RS78, p. 283]. Let $\mathcal{H}'$ be a Hilbert space and let $(M, \mathrm{d}\mu)$ be a measure space. Let $A(\cdot) : M \to \mathcal{L}^{s.a.}(\mathcal{H}')$ be a measurable function taking values in the space of self-adjoint operators (not necessarily bounded) on $\mathcal{H}'$ (with appropriate domains). Let $\mathcal{H} = \int_M^{\oplus} \mathcal{H}'$ and let $A = \int_M^{\oplus} A(m)\mathrm{d}\mu(m)$. For an element $f \in \mathcal{H}$, we denote its fibers as $f_m \in \mathcal{H}'$ so that

$$f = \int_M^{\oplus} f_m \,\mathrm{d}\mu(m). \tag{2.1}$$

We denote the resolution of the identity of A by $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ and of $A(m)$ by $\{E_m(\lambda)\}_{\lambda\in\mathbb{R}}$.

Lemma 2.3. *The resolutions of the identity also satisfy the natural decomposition*

$$E(\lambda) = \int_M^{\oplus} E_m(\lambda)\,\mathrm{d}\mu(m).$$

Proof. By standard functional calculus, we apply the characteristic function $\mathbb{1}_{(-\infty,\lambda_0]}$ to

$$\int_{\mathbb{R}} \lambda\,\mathrm{d}E(\lambda) = A = \int_M^{\oplus} A(m)\,\mathrm{d}\mu(m) = \int_M^{\oplus}\int_{\mathbb{R}} \lambda\,\mathrm{d}E_m(\lambda)\,\mathrm{d}\mu(m)$$

to obtain the assertion of the lemma (at the point λ_0). □

It is well-known that since all $A(m)$ are self-adjoint, so is A, with spectrum $\sigma(A)$ characterized as follows:

$$\lambda \in \sigma(A) \qquad \Leftrightarrow \qquad \forall \varepsilon > 0, \quad \mu\left(\{m \,:\, \sigma(A(m)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}\right) > 0. \tag{2.2}$$

An immediate consequence of this is:

$$\sigma(A) \subset \overline{\bigcup_m \sigma(A(m))}. \tag{2.3}$$

Let us mention some other important consequences. First, the following characterization of eigenvalues:

$$\mu\left(\{m \,:\, \lambda \text{ is an eigenvalue of } A(m)\}\right) > 0 \qquad \Rightarrow \qquad \lambda \text{ is an eigenvalue of } A. \tag{2.4}$$

Hence, if there exist real numbers E_j such that $\cup_m \sigma(A(m)) = \{E_1, \ldots, E_k\}$ with $\mu(\{m \,:\, \sigma(A(m)) = E_j\}) > 0$ for all j, then $\sigma(A) = \{E_1, \ldots, E_k\}$. If, on the other hand, for all $m \in M$ the spectrum of $A(m)$ is purely absolutely continuous then so is the spectrum of A.

Proposition 2.4. *Assume that the measure* $\mathrm{d}\mu$ *is the Borel measure associated to some given topology and that* M *is compact. Assume that* $\Sigma : M \to \mathrm{cl}(\mathbb{R}) = \{$*closed subsets in* $\mathbb{R}\}$ *given by* $m \mapsto \sigma(A(m)) \subset \mathbb{R}$ *is continuous (we take the Hausdorff distance on* $\mathrm{cl}(\mathbb{R})$*). Then* $\sigma(A) = \cup_m \sigma(A(m))$.

Proof. Since M is compact and Σ is continuous, we have that $\cup_m \sigma(A(m)) = \overline{\cup_m \sigma(A(m))}$. Hence, considering (2.3) we only need to prove that $\sigma(A) \supset \cup_m \sigma(A(m))$. Suppose that $\lambda \in \cup_m \sigma(A(m)) = \cup_m \Sigma(m)$. In particular there exists some $m_0 \in M$ such that $\lambda \in \Sigma(m_0)$. By the continuity of Σ, for each $\varepsilon > 0$ there exists an open neighborhood U of m_0 such that $\Sigma(m) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset$ for all $m \in U$, which, by (2.2), implies that $\lambda \in \sigma(A)$ (note that U has positive measure). □

3. Toy example

We view the flow operator (1.1) as fibered in the direction orthogonal to the flow. That is, we view it as $A = \int_{[0,1]}^{\oplus} A(m)\,\mathrm{d}m$ where $A(m)$ are one-dimensional flow operators, see (1.5). For small values of m, the (discrete) spectrum of the individual fibers concentrates near $\lambda = 0$, thereby making the analysis of the (continuous) spectrum of A near $\lambda = 0$ difficult. To improve our intuition, here we study a simplified problem.

Consider the following fibered problem. Let $(M, \mathrm{d}\mu) = ([0,1], \mathrm{d}m)$ with the usual Borel σ-algebra and suppose that for all $m \in [0,1]$ the spectrum of $A(m)$ in the energy band $I := (a,b) \subset \mathbb{R}$ is given by a single eigenvalue $\mathcal{E}(m)$ depending on m with multiplicity 1.

Suppose that $\mathcal{E} : [0,1] \to I$ is measurable (it actually has to be due to the definition of fibered operators). As before, we let $\mathcal{H} = \int_{[0,1]}^{\oplus} \mathcal{H}'$ and $A = \int_{[0,1]}^{\oplus} A(m)\mathrm{d}m$. We know that

- $\sigma(A) \cap I = \operatorname{ess\,ran}(\mathcal{E})$ and if $\mathcal{E}$ is continuous then ess ran may be replaced by ran (see Proposition 2.4).
- If $\mathcal{E}$ is constant on some open set U and equal to $\overline{\mathcal{E}}$ there, then $\overline{\mathcal{E}}$ is an eigenvalue of A (see (2.4)).

These two statements combined indicate that one can easily construct examples with eigenvalues embedded in the essential spectrum (or at its boundary). Hence even in this simple example, the absolute continuity of the spectrum of A depends on the nature of the function $\mathcal{E}$. Let us assume that there are no embedded eigenvalues:

Assumption A1. $\mathcal{E} \in C^1([0,1])$ and is not constant on sets of positive measure.

Definition 3.1. *Let $\lambda \in \operatorname{ran}(\mathcal{E})$ and denote $M_\lambda := \mathcal{E}^{-1}(\lambda) \subset [0,1]$. We say that λ is a regular value if for any $m \in M_\lambda$, $\mathcal{E}'(m) \neq 0$. We also define the set*

$$\sigma_{\mathrm{reg},I}(A) := \{\lambda \in I \ : \ \lambda \in \operatorname{ran}(\mathcal{E}) \text{ is a regular value}\}.$$

Lemma 3.2. *$\sigma_{\mathrm{reg},I}(A)$ is an open subset of I.*

Proof. The claim is almost trivial, as the condition of being a regular point is an open condition. We only note that any point in the set $I \cap \partial(\operatorname{ran}(\mathcal{E}))$ cannot be a regular value since I is open. $\square$

The central computation in the proof of the following theorem will appear in the proof of our main theorem in the next section, and therefore it is useful to understand it in this simplified toy model.

Theorem 3.3. *Suppose that $\mathcal{E}$ is as above and satisfies Assumption A1. Let $\lambda \in \sigma_{\mathrm{reg},I}(A)$. Then the DoS of A at λ satisfies the estimate*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f,g)_{\mathcal{H}} \right| \leq \left(\sum_{m \in M_\lambda} \frac{1}{|\mathcal{E}'(m)|} \right) \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}, \qquad \forall f,g \in \mathcal{H}. \tag{3.1}$$

Proof. Since $M = [0,1]$ is compact, $\mathcal{E} \in C^1$, and λ is a regular value, M_λ is a finite set and we denote its elements $M_\lambda = \{m_1, \dots, m_k\}$. Recall that for any $m \in [0,1]$, the spectrum of $A(m)$ in $I = (a,b)$ is made up of a single eigenvalue $\mathcal{E}(m)$. Hence if $\lambda \in I$ and $\mathcal{E}(m) \leq \lambda$ then the projection operator $E_m(\lambda)$ may be represented as $P_m(\mathcal{E}(m)) + E_m(a)$ where $P_m(\mathcal{E}(m))$ is the projection onto the eigenspace in $\mathcal{H}'$ (the mth "copy") corresponding to the eigenvalue $\mathcal{E}(m)$. If $\lambda \in I$ and $\mathcal{E}(m) > \lambda$ then the projection operator $E_m(\lambda)$ is equal to the projection operator $E_m(a)$. Letting $f, g \in \mathcal{H}$, with fibers $f_m, g_m \in \mathcal{H}'$ (as in (2.1)), we have

$$
\begin{aligned}
(E(\lambda)f, g)_{\mathcal{H}} &= \int_{[0,1]} (E_m(\lambda) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \\
&= \int_{\{m \,:\, \mathcal{E}(m) \le \lambda\}} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m + \int_{[0,1]} (E_m(a) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m.
\end{aligned}
$$

Differentiating in λ the second term on the right hand side is eliminated, and one is left with

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f, g)_{\mathcal{H}} &= \lim_{h \downarrow 0} \left[\frac{1}{h} \int_{\{m \,:\, \lambda < \mathcal{E}(m) \le \lambda + h\}} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right] \\
&= \sum_{i,\ \mathcal{E}'(m_i) > 0} \lim_{h \downarrow 0} \left[\frac{1}{h} \int_{(m_i, m_i + \frac{h}{\mathcal{E}'(m_i)}]} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right] \\
&\quad + \sum_{i,\ \mathcal{E}'(m_i) < 0} \lim_{h \downarrow 0} \left[\frac{1}{h} \int_{(m_i + \frac{h}{\mathcal{E}'(m_i)}, m_i]} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right].
\end{aligned}
$$

Making the change of variables $\eta_i = \frac{h}{|\mathcal{E}'(m_i)|}$ we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f, g)_{\mathcal{H}} &= \sum_{i,\ \mathcal{E}'(m_i) > 0} \frac{1}{\mathcal{E}'(m_i)} \lim_{\eta_i \downarrow 0} \left[\frac{1}{\eta_i} \int_{(m_i, m_i + \eta_i]} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right] \\
&\quad + \sum_{i,\ \mathcal{E}'(m_i) < 0} \frac{1}{|\mathcal{E}'(m_i)|} \lim_{\eta_i \downarrow 0} \left[\frac{1}{\eta_i} \int_{(m_i - \eta_i, m_i]} (P_m(\mathcal{E}(m)) f_m, g_m)_{\mathcal{H}'} \, \mathrm{d}m \right] \\
&= \sum_{i=1}^{k} \frac{1}{|\mathcal{E}'(m_i)|} (P_{m_i}(\mathcal{E}(m_i)) f_{m_i}, g_{m_i})_{\mathcal{H}'} \,. \qquad (3.2)
\end{aligned}
$$

Note that one must be careful if one of the m_i is 0 (resp. 1) and $\mathcal{E}'(0) < 0$ (resp. $\mathcal{E}'(1) > 0$) as then the above argument requires a slight adjustment. However the same conclusion holds. A simple use of the Cauchy-Schwartz inequality and the orthogonality properties of the projection operators leads to the desired estimate (3.1). □

4. Flow in an annulus: Proof of Theorem 1.1

4.1. **Description and assumptions.** We are now ready to study our main object of interest, a flow in an annulus. We consider a steady flow in an annulus $\mathcal{A} = [0,1] \times \mathbb{S}^1$. The variables in $[0,1]$ and $\mathbb{S}^1$ shall be denoted m and θ, respectively. For each $m \in (0,1]$ corresponds the self-adjoint operator

$$
A(m) = -i\varphi(m) \frac{\mathrm{d}}{\mathrm{d}\theta} : H^1(\mathbb{S}^1) \subset L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)
$$

generating a flow along the circle $\mathbb{S}^1$ with speed $\varphi(m) > 0$. Its spectrum is given by

$$
\sigma(A(m)) = \varphi(m)\mathbb{Z}.
$$

Furthermore, if φ is continuous on $[0,1]$ the spectrum of $A = \int_{[0,1]}^{\oplus} A(m)\,\mathrm{d}m$ is given by the expression

$$\sigma(A) = \left(\bigcup_{-k\in\mathbb{N}} \left[\varphi_{\max}k, \varphi_{\min}k\right] \right) \cup \left(\bigcup_{k\in\mathbb{N}\cup\{0\}} \left[\varphi_{\min}k, \varphi_{\max}k\right] \right)$$

where $\varphi_{\min} := \min_{[0,1]} \varphi(m)$ and $\varphi_{\max} := \max_{[0,1]} \varphi(m)$. For more details on how to derive this expression see for instance [Cox14]. We assume that the flow degenerates at the fiber $m = 0$, so that $\sigma(A) = \mathbb{R}$ (there is no spectral gap):

Assumption A2. Assume that there exists $\alpha > 0$ such that $\varphi(m) = m^{\alpha}$.

Remark 4.1. This assumption can be weakened, as only the behavior of φ near its zeros is important. For instance, all our arguments below could be modified to handle a case where there exist $m_0, c, \alpha > 0$ such that $\varphi(m) \sim cm^{\alpha}$ on $(0, m_0)$ and φ is continuous and positive on $[m_0, 1]$. This can then be extended to situations with several zeros, as long as φ vanishes like m^{α_i} for some $\alpha_i > 0$ near each zero. Furthermore, the requirement that φ be continuous can also be weakened. However these are technicalities that we do not pursue at present.

In the toy model presented in Section 3, to each energy level λ there corresponded finitely many fibers M_λ. Now the situation is more delicate: to each energy level correspond infinitely many fibers. Contributions to the spectrum at energy level $\lambda > 0$[2] will come from all $m_{\lambda,k} \in (0,1]$ such that $\lambda = km_{\lambda,k}^{\alpha}$ where $k \in \mathbb{N}$. That is,

$$m_{\lambda,k} = \left(\frac{\lambda}{k}\right)^{1/\alpha}, \qquad k \in \mathbb{N},\ \lambda > 0,$$

and conversely, each fiber $m \in (0,1]$ will contribute to the discrete energy levels

$$\lambda_{m,k} = km^{\alpha}, \qquad k \in \mathbb{N},\ m \in (0,1].$$

4.2. **Functional setting.** In order to prove Theorem 1.1, we must first obtain an estimate on the DoS as in (1.4). One of the crucial ingredients in such an estimate is the identification of a functional subspace $\mathcal{X} \subset L^2(\mathcal{A})$ on which the estimate holds. Here we define this space, denoted $\mathcal{H}_0^{s,\gamma}$, and discuss some of its properties.

We consider functions $f \in L^2(\mathcal{A})$ as fibered in m, as in (2.1):

$$f = \int_{[0,1]}^{\oplus} f_m\,\mathrm{d}m \qquad \text{with} \qquad f_m \in L^2(\mathbb{S}).$$

[2]Here we choose to consider positive energy levels $\lambda > 0$ and consequently $k \in \mathbb{N}$. We could have chosen $\lambda < 0$ and then $-k \in \mathbb{N}$.

This is equivalent to considering f in the space of functions $L^2\left([0,1];L^2(\mathbb{S})\right)$. In the θ variable we consider Sobolev regularity on the circle $\mathbb{S}$. For each function $f_m \in L^2(\mathbb{S})$, we define the Fourier series of f_m by

$$\widehat{f_m}(k) := \int_{[0,1]} f_m(\theta) e^{-2\pi i k\theta}\,\mathrm{d}\theta.$$

Since we are ultimately interested in the rate of convergence to the average, it is sensible to work with functions whose average along each fiber is zero:

$$L^2_0(\mathbb{S}) = \left\{g \in L^2(\mathbb{S}) \,:\, \widehat{g}(0) = 0\right\}.$$

Remark 4.2. There is no loss of generality in considering $L^2_0(\mathbb{S})$ on each fiber. Indeed, for a general function $f \in L^2(\mathcal{A})$ each of the fibers $(f - Pf)_m$ of $f - Pf$ belongs to $L^2_0(\mathbb{S})$ (here P is as defined in (1.2)). As Pf is invariant under the flow of the Hamiltonian, we can safely replace f by $f - Pf$ in all the following.

We consider now standard homogeneous Sobolev spaces on $L^2_0(\mathbb{S})$. These are defined as

$$\dot{H}^\gamma := \left\{ g \in L^2_0(\mathbb{S}) \,:\, \|g\|^2_{\dot{H}^\gamma} := \sum_{k\in\mathbb{Z}} |k|^{2\gamma} |\widehat{g}(k)|^2 < +\infty \right\}, \qquad \gamma \geq 0.$$

We associate to these spaces the norm $\|\cdot\|_{\dot{H}^\gamma}$.

Regarding the regularity with respect to the m variable in the closed interval $[0,1]$, we consider f as a function of $L^2\left([0,1];\dot{H}^\gamma\right)$. Such spaces of functions and associated fractional Sobolev regularity have been extensively studied by Lions and Magenes in [LM72]. In the following we use the framework and the results of [Sim90] (see Theorem 10.2 in [LM72] for the equivalence of the viewpoints, as Lions and Magenes define fractional Sobolev spaces on an interval by interpolation spaces).

We define then, for $s \in (0,1)$ and $\gamma \geq 0$, the space

$$\dot{\mathcal{H}}^{s,\gamma} := \left\{ f \in L^2\left([0,1];\dot{H}^\gamma\right) \,:\, \frac{\|f_m - f_{m'}\|_{\dot{H}^\gamma}}{|m-m'|^{1/2+s}} \in L^2\,[0,1]\times[0,1]) \right\}.$$

This is in fact $\dot{\mathcal{H}}^{s,\gamma} = W^{s,2}\left([0,1];\dot{H}^\gamma\right)$ in the framework of [Sim90]. We associate to this space the usual norm defined by

$$\|f\|^2_{s,\gamma} = \|f\|^2_{L^2\left([0,1];\dot{H}^\gamma\right)} + \iint_{[0,1]\times[0,1]} \frac{\|f_m - f_{m'}\|^2_{\dot{H}^\gamma}}{|m-m'|^{1+2s}}\,\mathrm{d}m\,\mathrm{d}m'. \tag{4.1}$$

One important result is the following Sobolev embedding-type theorem:

Proposition 4.3. *Let $s > 1/2$. Then any $f \in \dot{\mathcal{H}}^{s,\gamma}$ is $(s-1/2)$-Hölder continuous with respect to the m variable. That is, there is a constant $C(s)$ (independent of f and γ) such that*

$$\|f_m - f_{m'}\|_{\dot{H}^\gamma} \leq C(s)|m-m'|^{s-1/2}, \qquad \forall m,m' \in [0,1],\, \forall f \in \dot{\mathcal{H}}^{s,\gamma}. \tag{4.2}$$

For a proof of this result, we refer to the paper of Simon [Sim90] and in particular Corollary 26 therein as $\dot{\mathcal{H}}^{s,\gamma} = W^{s,2}\left([0,1];\dot{H}^\gamma\right)$ and Hölder continuity is called Lipschitz continuity.

We may now define the space in which we will work:

Definition 4.4. *Let $s > 1/2, \gamma \geq 0$. We define the subspace $\dot{\mathcal{H}}_0^{s,\gamma} \subset \dot{\mathcal{H}}^{s,\gamma}$ to be the set of functions that vanish along $\{m = 0\}$:*

$$\dot{\mathcal{H}}_0^{s,\gamma} := \left\{ f \in \dot{\mathcal{H}}^{s,\gamma} \,:\, f_0 = 0 \right\}. \tag{4.3}$$

We equip this subspace with the norm $\|\cdot\|_{s,\gamma}$ defined above in (4.1).

We now make some important observations regarding this L^2 subspace. Thanks to the Hölder continuity result of Proposition 4.3, functions in $\dot{\mathcal{H}}_0^{s,\gamma}$ decay as $m \downarrow 0$:

Proposition 4.5. *For all $f \in \dot{\mathcal{H}}_0^{s,\gamma}$, there holds*

$$\|f_m\|_{\dot{H}^\gamma} \leq C(s)\,\|f\|_{s,\gamma}\, m^{s-1/2}, \qquad \forall m \in [0,1], \tag{4.4}$$

where $C(s) > 0$ is the same constant appearing in (4.2) *and which does not depend on γ.*

Proof. This follows immediately from the inequality (4.2) together with the fact that $f_0 = 0$ since $f \in \dot{\mathcal{H}}_0^{s,\gamma}$. □

Next, we state a result on the Hölder regularity of the Fourier coefficients of functions in $\dot{\mathcal{H}}^{s,\gamma}$, which is just a corollary of the Hölder continuity of f stated in Proposition 4.3:

Proposition 4.6. *For any $f \in \dot{\mathcal{H}}_0^{s,\gamma}$, there holds*

$$\left|\widehat{f}_m(k) - \widehat{f}_{m'}(k)\right| \leq C\|f\|_{s,\gamma}|k|^{-\gamma}|m-m'|^{s-1/2}, \qquad \forall m, m' \in [0,1],\, \forall k \in \mathbb{Z}, \tag{4.5}$$

where the constant $C > 0$ does not depend on f, k or m.

Proof. We readily compute

$$\begin{aligned}\left|\widehat{f}_m(k) - \widehat{f}_{m'}(k)\right|^2 &= |k|^{-2\gamma}|k|^{2\gamma}\left|\widehat{f}_m(k) - \widehat{f}_{m'}(k)\right|^2 \\ &\leq |k|^{-2\gamma}\sum_{k'\in\mathbb{Z}}|k'|^{2\gamma}\left|\widehat{f}_m(k') - \widehat{f}_{m'}(k')\right|^2 \leq |k|^{-2\gamma}\|f_m - f_{m'}\|_{\dot{H}^\gamma}^2.\end{aligned}$$

By the Hölder continuity (4.2) of f_m in the $\dot{H}^\gamma$ norm, we obtain (4.5). □

4.3. **Density of states estimate.** The main ingredient in the proof of Theorem 1.1 is an estimate of the DoS near 0. This is stated as follows:

Theorem 4.7. *Let $A = -i\varphi(m)\frac{\partial}{\partial\theta}$ satisfy Assumption A2. Further assume that $s > 1/2$ and $\gamma \geq 0$ satisfy the constraint*

$$\gamma + s/\alpha > 1/2. \tag{4.6}$$

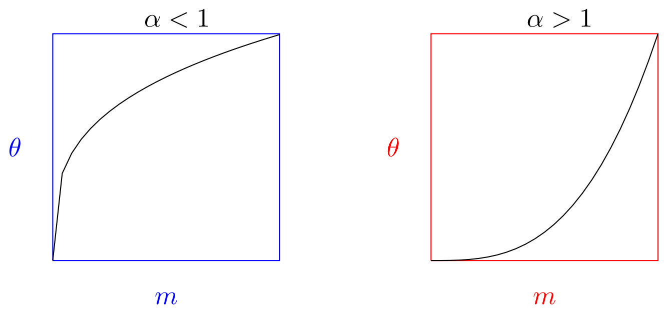
Then there exists $r > 0$ such that the DoS of A satisfies

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)f,g)_{L^2(\mathcal{A})}\right| \leq C|\lambda|^{\frac{2s}{\alpha}-1}\left\|f\right\|_{s,\gamma}\left\|g\right\|_{s,\gamma}, \qquad \forall\lambda \in (-r,r)\setminus\{0\},\ \forall f,g \in \dot{\mathcal{H}}_0^{s,\gamma},$$

where $C > 0$ does not depend on f, g or λ.

Remark 4.8. 1. In the case $\alpha \leq 1$ the constraint (4.6) is satisfied for all $s > 1/2$ with $\gamma = 0$. This means that for $\alpha \leq 1$, the subspace $\dot{\mathcal{H}}_0^{s,0}$ is sufficient to get the estimate of the DoS.

2. In the case $\alpha > 1$ however, the constraint (4.6) is stronger and working in the subspace $\dot{\mathcal{H}}_0^{s,0}$ is not sufficient to obtain estimates of the DoS. This is mainly due to the fact that the eigenvalues $\lambda_{m,k} = km^\alpha$ concentrate at 0 faster as $m \to 0$ and $k \to \infty$. To balance that effect, more regularity in the θ variable is required, as the equation $\lambda = \lambda_{m,k}$ allows to trade control of high frequencies in smallness in m.



Proof of Theorem 4.7. As in the case of the toy model presented in Section 3, we write

$$\begin{aligned}(E(\lambda)f,g)_{L^2(\mathcal{A})} &= \int_{[0,1]} (E_m(\lambda)f_m, g_m)_{L^2(\mathbb{S})}\,\mathrm{d}m \\ &= \int_{[0,1]} \sum_{\substack{k\in\mathbb{Z} \\ \lambda_{m,k}\leq\lambda}} (P_m(\lambda_{m,k})f_m, g_m)_{L^2(\mathbb{S})}\,\mathrm{d}m\end{aligned}$$

where $P_m(\lambda_{m,k})$ is the projection on the Fourier coefficient $\widehat{f_m}(k)$ of f_m.

As before, we compute the DoS starting from the definition of a derivative, i.e. the limit of $\frac{1}{h}\left\{(E(\lambda+h)f,g)_{L^2(\mathcal{A})} - (E(\lambda)f,g)_{L^2(\mathcal{A})}\right\}$ as $h \to 0$. We define the energy band

$$\mathcal{B}(\lambda,m,h) = \{k \in \mathbb{Z} \,:\, \lambda < km^\alpha \leq \lambda + h\}$$

and have that

$$\begin{aligned}
&\frac{1}{h}\left\{(E(\lambda+h)f,g)_{L^2(\mathcal{A})}-(E(\lambda)f,g)_{L^2(\mathcal{A})}\right\}\\
&\quad=\frac{1}{h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\left(P_m(\lambda_{m,k})f_m,g_m\right)_{L^2(\mathbb{S})}\,\mathrm{d}m\\
&\quad=\frac{1}{h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m.
\end{aligned}$$

We would now like to commute the integration and summation, to obtain the sum of integrals over small subintervals of $[0,1]$, much like in the computations in (3.2), in order to then take the limit $h\to 0$. However, for fixed h the sets $\mathcal{B}(\lambda,m,h)$ will contain arbitrarily many elements as $m\to 0$. Moreover, as $m\to 0$ the preimages of $\lambda<km^\alpha\le\lambda+h$ in $m\in[0,1]$ are no longer disjoint, resulting in many redundant integrations. Indeed, intervals $(\lambda/m^\alpha,(\lambda+h)/m^\alpha]$ are of size larger than one if $m<h^{1/\alpha}$, and then may contain more than one integer. Our strategy is to split the previous integral into two parts

$$\frac{1}{h}\int_{[0,1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m=\mathcal{I}(h)+\mathcal{R}(h)$$

where

$$\mathcal{R}(h)=\frac{1}{h}\int_{[0,h^{1/\alpha})}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m.$$

and

$$\mathcal{I}(h)=\frac{1}{h}\int_{[h^{1/\alpha},1]}\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m$$

1. The term $\mathcal{R}(h)$. We show that $\lim_{h\to 0}\mathcal{R}(h)=0$. Using the inequalities

$$\left|\sum_{k\in\mathcal{B}(\lambda,m,h)}\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right|\le\sum_{k\in\mathcal{B}(\lambda,m,h)}\left|\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right|\le\sum_{k>\lambda/m^\alpha}\left|\widehat{f_m}(k)\overline{\widehat{g_m}(k)}\right|$$

and taking $f,g\in\dot{\mathcal{H}}^{s,\gamma}_0$ we have

$$\begin{aligned}
|\mathcal{R}(h)|&\le\frac{1}{h}\int_{[0,h^{1/\alpha})}\sum_{k>\lambda/m^\alpha}|k|^{-2\gamma}\left(|k|^\gamma\left|\widehat{f_m}(k)\right|\right)\left(|k|^\gamma\left|\overline{\widehat{g_m}(k)}\right|\right)\,\mathrm{d}m\\
&\le\frac{1}{h}\int_{[0,h^{1/\alpha})}\sup_{k>\lambda/m^\alpha}\left(|k|^{-2\gamma}\right)\|f_m\|_{\dot{H}^\gamma}\|g_m\|_{\dot{H}^\gamma}\,\mathrm{d}m,
\end{aligned}$$

using the Hölder inequality for the last line. We use now inequality (4.4) that gives decay as $m\to 0$ for the norm $\|f_m\|_{\dot{H}^\gamma}$ and compute then

$$\begin{aligned}
|\mathcal{R}(h)|&\le\frac{1}{h}\int_{[0,h^{1/\alpha})}\left(\frac{\lambda}{m^\alpha}\right)^{-2\gamma}\|f_m\|_{\dot{H}^\gamma}\|g_m\|_{\dot{H}^\gamma}\,\mathrm{d}m\\
&\le C\|f\|_{s,\gamma}\|g\|_{s,\gamma}\frac{\lambda^{-2\gamma}}{h}\int_{[0,h^{1/\alpha})}m^{2\alpha\gamma+2(s-1/2)}\,\mathrm{d}m\\
&=C\|f\|_{s,\gamma}\|g\|_{s,\gamma}\lambda^{-2\gamma}h^{2\gamma+2s/\alpha-1}.
\end{aligned}$$

As soon as the constraint (4.6) on s and γ is satisfied there holds

$$\lim_{h\to 0} \mathcal{R}(h) = 0.$$

2. The term $\mathcal{I}(h)$. Whenever $m \geq h^\alpha$, the energy band $\mathcal{B}(\lambda, m, h)$ contains at most one integer. For $\mathcal{B}(\lambda, m, h)$ to contain one integer, by definition m has to be in an interval of the form $((\lambda/k')^{1/\alpha}, ((\lambda+h)/k')^{1/\alpha}]$ for some $k' \in \mathbb{N}$. As $m \in [h^{1/\alpha}, 1]$, this implies for the bounds of such an interval that

$$h^{1/\alpha} \leq (\lambda/k')^{1/\alpha} \quad \text{and} \quad ((\lambda+h)/k')^{1/\alpha} \leq 1.$$

The integer k' has then to satisfy the bounds

$$N(\lambda, h) := \lfloor \lambda/h \rfloor \geq k' \geq k_0 := \max(1, \lfloor \lambda \rfloor),$$

reminding that $k' \geq 1$. We write then

$$\begin{aligned}
\mathcal{I}(h) &= \frac{1}{h} \int_{[h^{1/\alpha},1]} \sum_{k\in\mathcal{B}(\lambda,m,h)} \widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m \\
&= \frac{1}{h} \sum_{N(\lambda,h)\geq k'\geq k_0} \int_{((\lambda/k')^{1/\alpha},((\lambda+h)/k')^{1/\alpha}]} \sum_{k\in\mathcal{B}(\lambda,m,h)} \widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m \\
&= \frac{1}{h} \sum_{N(\lambda,h)\geq k'\geq k_0} \int_{((\lambda/k')^{1/\alpha},((\lambda+h)/k')^{1/\alpha}]} \widehat{f_m}(k')\overline{\widehat{g_m}(k')}\,\mathrm{d}m.
\end{aligned}$$

Next, we use the same kind of computation as in the proof of Theorem 3.3, where the prefactor $\frac{1}{\mathcal{E}'(m)}$ appears. In the current computation, the function $\mathcal{E}(m)$ becomes $k\varphi(m) = km^\alpha$ so that $\mathcal{E}'(m)$ becomes $k\alpha m^{\alpha-1}$. Substituting $\left(\frac{\lambda}{k}\right)^{1/\alpha}$ for m we find that the prefactor should be $\frac{1}{\alpha k}\left(\frac{\lambda}{k}\right)^{1/\alpha-1}$. Furthermore, we use Proposition 4.6 which ensures some uniform regularity for the Fourier coefficients and allows us to make pointwise (in m) evaluations. This leads to

$$\lim_{h\to 0} \frac{1}{h} \int_{((\lambda/k)^{1/\alpha},((\lambda+h)/k)^{1/\alpha}]} \widehat{f_m}(k)\overline{\widehat{g_m}(k)}\,\mathrm{d}m = \frac{1}{\alpha k}\left(\frac{\lambda}{k}\right)^{1/\alpha-1} \widehat{f}_{(\lambda/k)^{1/\alpha}}(k)\overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)}$$

which gives

$$\lim_{h\to 0} \mathcal{I}(h) = \sum_{k\geq k_0} \frac{1}{\alpha k}\left(\frac{\lambda}{k}\right)^{1/\alpha-1} \widehat{f}_{(\lambda/k)^{1/\alpha}}(k)\,\overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)}.$$

by uniform boundedness. To conclude we use inequality (4.5) and the fact that f_0 is trivial in $\dot{H}^\gamma$ to obtain

$$\left|\widehat{f}_{(\lambda/k)^{1/\alpha}}(k)\right| \leq C\|f\|_{s,\gamma}|k|^{-\gamma}\left|(\lambda/k)^{1/\alpha}\right|^{s-1/2}$$

with the constant $C > 0$ independent of k and λ. There holds now for $\lim_{h\to 0}\mathcal{I}(h)$ the bound

$$\begin{aligned}\left|\lim_{h\to 0}\mathcal{I}(h)\right| &= \frac{1}{\alpha}\left|\sum_{k\geq k_0}\frac{1}{k}\left(\frac{\lambda}{k}\right)^{1/\alpha-1}\widehat{f}_{(\lambda/k)^{1/\alpha}}(k)\overline{\widehat{g}_{(\lambda/k)^{1/\alpha}}(k)}\right| \\ &\leq C\,\|f\|_{s,\gamma}\,\|g\|_{s,\gamma}\,\frac{\lambda^{1/\alpha-1}}{\alpha}\sum_{k\geq k_0}k^{-1/\alpha-2\gamma}\left(\frac{\lambda}{k}\right)^{2(s-1/2)/\alpha} \\ &= C\,\|f\|_{s,\gamma}\,\|g\|_{s,\gamma}\,\frac{\lambda^{2s/\alpha-1}}{\alpha}\sum_{k\geq k_0}k^{-(1/\alpha+2\gamma+2(s-1/2)/\alpha)} \\ &\leq C\,\|f\|_{s,\gamma}\,\|g\|_{s,\gamma}\,\frac{\lambda^{2s/\alpha-1}}{\alpha}\sum_{k\geq 1}k^{-(2\gamma+2s/\alpha)}\end{aligned}$$

where we use $k_0 \geq 1$ in the last inequality. The sum over k is then finite if the constraint (4.6) on s and γ is satisfied. This completes the proof of Theorem 4.7. □

4.4. **Uniform ergodic theorem: proof of Theorem 1.1.** Finally we are ready to prove Theorem 1.1. In [BAM19] we prove the following result:

Theorem 4.9 (Corollary 1.5 in [BAM19]). *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be self-adjoint and assume that there exist a Banach subspace $\mathcal{X} \subset \mathcal{H}$ which is dense in $\mathcal{H}$ in the topology of $\mathcal{H}$ that is continuously embedded in $\mathcal{H}$ (and therefore the norm $\|\cdot\|_{\mathcal{X}}$ is stronger than the norm $\|\cdot\|_{\mathcal{H}}$), and positive numbers $C, p, r > 0$ such that:*

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(E(\lambda)f, g\right)_{\mathcal{H}}\right| \leq C|\lambda|^{p-1}\|f\|_{\mathcal{X}}\|g\|_{\mathcal{X}}, \qquad \forall f, g \in \mathcal{X},\ \forall\lambda \in (-r, r)\setminus\{0\}.$$

Then the operators $P^T f := \frac{1}{2T}\int_{-T}^{T} e^{itA} f\,\mathrm{d}t$ and $P :=$ orthogonal projection of $\mathcal{H}$ onto $\ker A$ satisfy

$$\|P^T - P\|_{\mathcal{X}\to\mathcal{H}} \leq C(p)T^{-\frac{p}{2+p}}, \qquad \forall T > 1.$$

This result readily allows us to prove our main theorem:

Proof of Theorem 1.1. We can invoke Theorem 4.9. From Theorem 4.7 we know that the bound on the density of states is satisfied with

- $\mathcal{H} = L^2(\mathcal{A})$,
- $\mathcal{X} = \dot{\mathcal{H}}_0^{s,\gamma}$,
- $p = \frac{2s}{\alpha}$.

Observing that $\frac{p}{2+p} = \frac{\frac{2s}{\alpha}}{\frac{2s}{\alpha}+2} = \frac{s}{s+\alpha}$ we deduce that

$$\|P^T - P\|_{\dot{\mathcal{H}}_0^{s,\gamma}\to L^2(\mathcal{A})} \leq CT^{-\frac{s}{s+\alpha}}, \qquad \forall T > 1,$$

where $s > 1/2$ and $\gamma \geq 0$ satisfy $\gamma + s/\alpha > 1/2$, and where $C > 0$ is a constant that does not depend on T. This completes the proof. □

We finish with a few remarks:

Remark 4.10. 1. Note that the rate does not depend on γ. Note also that, even if there is a threshold regarding α less or greater than 1 in the regularity constraint (4.6) (see Remark 4.8), there is no such thing regarding the exponent $-\frac{s}{s+\alpha}$.

2. As $\alpha \downarrow 0$ the exponent tends to 1 which is the rate of convergence in the case of a spectral gap. Intuitively, smaller α brings us closer to the case of a spectral gap in the sense that there are "fewer" slow trajectories.

3. We also observe that $\frac{s}{s+\alpha} \to 1$ as $s \to +\infty$. Informally, this expresses the fact that "infinite regularity" in the m variable gives the rate of convergence of the spectral gap case.

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