WEAK POINCARÉ INEQUALITIES IN THE ABSENCE OF SPECTRAL GAPS

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ABSTRACT. For generators of Markov semigroups which lack a spectral gap, it is shown how bounds on the density of states near zero lead to a so-called "weak Poincaré inequality" (WPI), originally introduced by Liggett [Ann. Probab., 1991]. Applications to general classes of constant coefficient pseudodifferential operators are studied. Particular examples are the heat semigroup and the semigroup generated by the fractional Laplacian in the whole space, where the optimal decay rates are recovered. Moreover, the classical Nash inequality appears as a special case of the WPI for the heat semigroup.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

In this note we study how the well-known equivalence between spectral gaps, Poincaré inequalities and exponential rates of decay to equilibrium extends to systems which lack a spectral gap but have a bounded density of states near 0. Our main result relies solely on our ability to "differentiate" the resolution of the identity of a given operator. It is thus quite general, and covers important examples such as Markov semigroups.

Our setup is as follows: we let M be a manifold with Borel measure $\mathrm{d}\mu$, $\mathcal{H} = L^2(M, \mathrm{d}\mu; \mathbb{R})$ equipped with scalar product $(\cdot, \cdot)_{\mathcal{H}}$. We assume that $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ is a self-adjoint, non-negative operator, so that $-H$ is the infinitesimal generator of a Markov semigroup $(P_t)_{t \geq 0}$, whose invariant measure is $\mathrm{d}\mu$, i.e. for every u that is bounded and non-negative $\int_M P_t u \, \mathrm{d}\mu = \int_M u \, \mathrm{d}\mu$ for any $t \geq 0$. Let $\{E(\lambda)\}_{\lambda \geq 0}$ be the resolution of the identity of H

and let the associated Dirichlet form be

$$\mathcal{E}(u) := \int_M (H^{1/2}u)^2 \, \mathrm{d}\mu.$$

As stated above, instead of assuming a spectral gap, we assume the opposite: H has continuous spectrum in a neighborhood of 0 (and 0 itself is possibly an eigenvalue). We show that an appropriate estimate of the density of the spectrum near 0 leads to a weaker version of the Poincaré inequality (also known as a weak Poincaré inequality, defined below in Definition 1.3). This, in turn, leads to an algebraic decay rate for the associated semigroup.

In this paper we employ the following definition for the variance of a given function $u \in \mathcal{H}$:

$$\mathrm{Var}(u) := \int_M (u - E(\{0\})u)^2 \, \mathrm{d}\mu$$

where $E(\{0\})$ is the projection onto the kernel of H. In the case where the kernel only consists of constant functions and μ is a probability measure, this definition coincides with the standard definition, see [3, §4.2.1]. We discuss the significance of the resolution of the identity of H (and in particular the projection onto its kernel) and its relationship with functional inequalities and decay rates below in Section 2.3.

We can now recall the classical Poincaré inequality (again, see [3, §4.2.1]):

Definition 1.1 (Poincaré Inequality).   *We say that H satisfies a* Poincaré inequality *if there exists $C > 0$ such that*

$$\mathrm{Var}(u) \le C\mathcal{E}(u), \qquad \forall u \in D(\mathcal{E}),$$

where C does not depend on u.

Remark 1.2.   The topology of $D(\mathcal{E})$ is the graph norm topology generated by $\|\cdot\|_{\mathcal{H}}^2 + \mathcal{E}(\cdot)$, see [3, §3.1.4].

The definition of a "weak Poincaré inequality" is somewhat ambiguous. This is addressed in further detail in Section 2.3 below. We adopt the following definition, motivated by Liggett [13, Equation (2.3)]:

Definition 1.3 (Weak Poincaré Inequality).   *Let $\Phi : \mathcal{H} \to [0, \infty]$ satisfy $\Phi(u) < \infty$ on a dense subset of $D(\mathcal{E})$. Let $p \in (1, \infty)$. We say that H satisfies a (Φ, p)-weak Poincaré inequality ((Φ, p)-WPI) if there exists $C > 0$ such that*

$$\mathrm{Var}(u) \le C\mathcal{E}(u)^{1/p}\Phi(u)^{1/q}, \qquad \forall u \in D(\mathcal{E}), \tag{1.1}$$

where C does not depend on u and where $1/p + 1/q = 1$.

Remark 1.4.   Note that (1.1) is meaningful only on a dense subset of $D(\mathcal{E})$ where $\Phi < +\infty$.

1.1. **The Hilbertian case.** We start our discussion by considering the purely Hilbertian case, i.e. we consider generators with density of states that are defined on subspaces which respect the Hilbert structure of $\mathcal{H}$, such as Sobolev spaces or weighted spaces. Our basic assumption is:

Assumption A1.   There exists a dense subspace $\mathcal{X} \subset \mathcal{H}$ such that

(1) $\mathcal{X} \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ (in the topology of $D(\mathcal{E})$),

(2) for some constants $r > 0$, $C_1 > 0$ and $\alpha > -1$,

the mapping $\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u, v)_{\mathcal{H}}$ is continuous on $(0, r)$ for every $u, v \in \mathcal{X}$ and satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u, v)_{\mathcal{H}} \right| \leq C_1 \lambda^{\alpha} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}, \qquad \forall u, v \in \mathcal{X}, \forall \lambda \in (0, r). \tag{1.2}$$

Remark 1.5. We refer to the bilinear form $\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)\cdot, \cdot)_{\mathcal{H}}$ as the density of states (DoS) of H at λ. Note that if the DoS satisfies a bound as in (1.2) and $\mathcal{X}$ has a norm compatible with (and stronger than) the norm on $\mathcal{H}$ then it induces an operator $\mathcal{X} \to \mathcal{X}^*$ by the Riesz representation theorem.

We can finally state our main results on how (1.2) leads to a (Φ, p)-WPI (Theorem 1.6) and, in turn, an explicit rate of decay (Theorem 1.7). Theorem 1.6 will be further generalized below in Theorem 1.9, and then again in Proposition 1.13 where a precise constant in the WPI is obtained. The decay rates presented in Theorem 1.7 apply to the Markov semigroup generated by H.

Theorem 1.6. *If Assumption A1 holds then H satisfies a (Φ, p)-weak Poincaré inequality with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ (and $\Phi(u) = +\infty$ if $u \in \mathcal{H} \setminus \mathcal{X}$) and $p = \frac{2+\alpha}{1+\alpha}$.*

Theorem 1.7. *Let Assumption A1 hold. Let $u \in \mathcal{X}$ and suppose that there exist $C_2 = C_2(u) \geq 0$ and $\beta \in \mathbb{R}$, such that the Markov semigroup satisfies*

$$\|P_t u\|_{\mathcal{X}}^2 \leq \|u\|_{\mathcal{X}}^2 + C_2 t^{\beta}, \qquad \forall t \geq 0. \tag{1.3}$$

Then

$$\mathrm{Var}(P_t u) \leq \left(\mathrm{Var}(u)^{\frac{-1}{1+\alpha}} + C_3 \int_0^t (\|u\|_{\mathcal{X}}^2 + C_2 s^{\beta})^{\frac{-1}{1+\alpha}} \, \mathrm{d}s \right)^{-(1+\alpha)} \tag{1.4}$$

where C_3 is given explicitly (and only depends on α, C_1). In particular, $\mathrm{Var}(P_t u)$ satisfies the following decay rates as $t \to +\infty$:

$$\mathrm{Var}(P_t u) \leq \begin{cases} O((\log t)^{-(1+\alpha)}) & \beta = 1 + \alpha. \\ O(t^{\beta - (1+\alpha)}) & 0 < \beta < 1 + \alpha. \\ O(t^{-(1+\alpha)}) & C_2 = 0 \text{ or } \beta \leq 0. \end{cases}$$

Remark 1.8. 1. The choice of space $\mathcal{X}$ is motivated by (1.3): it is beneficial to choose $\mathcal{X}$ that is invariant under the Markov semigroup (i.e., if $u \in \mathcal{X}$ then $P_t u \in \mathcal{X}$ for all $t \geq 0$).

2. Clearly $C_2(u)$ is subject to quadratic scaling, for example it can be $C\|u\|_{\mathcal{H}}^2$ or $C\|u\|_{\mathcal{X}}^2$, but the explicit form is not important.

1.2. **A generalized theorem: departing from the Hilbert structure.** Theorems 1.6 and 1.7 demonstrate how estimates on the density of states near 0 imply a weak Poincaré inequality and a rate of decay to equilibrium. However it is not essential to restrict oneself to a subspace $\mathcal{X}$. In fact, it is often desirable to deal with functional spaces that are not contained in $\mathcal{H}$, as it may provide improved estimates and decay rates. In particular, this makes sense when the operator in question is the generator of a Markov semigroup, and acts

on a range of spaces simultaneously. Hence we replace Assumption A1 by a more general one:

Assumption A2. There exist Banach spaces $\mathcal{X}, \mathcal{Y}$ of functions on M, a constant $r > 0$ and a function $\psi_{\mathcal{X},\mathcal{Y}} \in L^1(0,r)$ that is strictly positive a.e. on $(0,r)$, such that

(1) $\mathcal{X} \cap \mathcal{Y} \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ (in the topology of $D(\mathcal{E})$).
(2) The mapping $\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u,v)_{\mathcal{H}}$ is continuous on $(0,r)$ for every $u \in \mathcal{X} \cap \mathcal{H}$ and $v \in \mathcal{Y} \cap \mathcal{H}$ and satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)u,v)_{\mathcal{H}} \right| \leq \psi_{\mathcal{X},\mathcal{Y}}(\lambda) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}, \qquad \forall \lambda \in (0,r). \tag{1.5}$$

We can now state the following more general theorem:

Theorem 1.9. *Let the conditions of Assumption A2 hold, and define* $\Psi_{\mathcal{X},\mathcal{Y}}(\rho) = \int_0^\rho \psi_{\mathcal{X},\mathcal{Y}}(\lambda)\,\mathrm{d}\lambda$, $\rho \in (0,r)$. *Then:*

a. There exists $K_0 \in (0,1)$ such that the following functional inequality holds:

$$(1-K)\Psi_{\mathcal{X},\mathcal{Y}}^{-1}\left(K \frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}} \right) \mathrm{Var}(u) \leq \mathcal{E}(u), \qquad \forall K \in (0,K_0), \forall u \in D(\mathcal{E}) \tag{1.6}$$

where $\|u\|_{\mathcal{X}} = +\infty$ if $u \notin \mathcal{X}$ and similarly for $\mathcal{Y}$.

b. If $\mathcal{X} = \mathcal{Y}$ and $\psi_{\mathcal{X},\mathcal{Y}}(\lambda) = C_1 \lambda^\alpha$, $\alpha > -1$, the estimate (1.6) reduces to the (Φ,p)-WPI as in Definition 1.3 with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ and $p = \frac{\alpha+2}{\alpha+1}$.

c. If, in addition, $\mathcal{X} = \mathcal{Y} \subset \mathcal{H}$ then we obtain Theorem 1.6.

Remark 1.10. The inequality (1.6) can be viewed as an implicit form of the weak Poincaré inequality. Note that setting $K = 0$ (which is excluded in the theorem) leads to the Poincaré inequality.

The power of this result is demonstrated in the following corollary, where the celebrated Nash inequality is obtained as a simple consequence. This simple derivation is discussed in Remark 4.2 below.

Corollary 1.11 (Nash inequality). *When $H = -\Delta : H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and $\mathcal{Y} = \mathcal{X} = L^1(\mathbb{R}^d)$ the inequality (1.6) is precisely Nash's inequality [15]:*

$$\|u\|_{L^2}^2 \leq C \left(\|\nabla u\|_{L^2}^2 \right)^{\frac{d}{d+2}} \left(\|u\|_{L^1}^2 \right)^{\frac{2}{d+2}}, \qquad \forall u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d),$$

where $C > 0$ does not depend on u. Furthermore, using Proposition 1.13 (below) an explicit constant may be computed to yield $C = \left(\frac{|\mathbb{S}^{d-1}|}{2} \right)^{\frac{2}{2+d}} \frac{2+d}{d}$.

Proof. The (simple) proof of this corollary is done by applying our results to the heat semigroup. More details are provided in the examples (Section 4), in particular see Remark 4.2. $\square$

Remark 1.12. The requirement that $\psi_{\mathcal{X},\mathcal{Y}}$ is strictly positive a.e. on $(0,r)$, for some $r > 0$ (perhaps very small), is quite natural as we are interested in operators that lack a spectral gap. However, one can easily generalize our result even if that is not the case by defining

$$\Psi_{\mathcal{X},\mathcal{Y}}^{-1}(y) = \sup\left\{ x \in (0,r) \mid \Psi_{\mathcal{X},\mathcal{Y}}(x) \leq y \right\}.$$

1.3. **Precise constants.** Under additional mild assumptions one can improve Theorem 1.9 by replacing the inequality (1.6) which contains an arbitrary constant K with an inequality that has an explicit constant. The question of how far this constant is from being sharp is the topic of ongoing research.

Proposition 1.13. *Let the conditions of Assumption A2 hold. Assume in addition that $\psi_{\mathcal{X},\mathcal{Y}}$ can be extended to a continuous function on $(0, R)$, where $R \in [r, +\infty]$ is such that if $\Psi_{\mathcal{X},\mathcal{Y}}(\rho) := \int_0^\rho \psi_{\mathcal{X},\mathcal{Y}}(\lambda) \, \mathrm{d}\lambda$, $\rho \in (0, R)$ and*

$$g_{\mathcal{X},\mathcal{Y}}(\rho) := \Psi_{\mathcal{X},\mathcal{Y}}(\rho) + \rho\psi_{\mathcal{X},\mathcal{Y}}(\rho)$$

then g is non-decreasing and $\lim_{\rho \to 0^+} g_{\mathcal{X},\mathcal{Y}}(\rho) = 0$, $\lim_{\rho \to R^-} g_{\mathcal{X},\mathcal{Y}}(\rho) = +\infty$. Then:

a. The following functional inequality holds:

$$\left(g_{\mathcal{X},\mathcal{Y}}^{-1}\left(\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}\right)\right)^2 \psi_{\mathcal{X},\mathcal{Y}}\left(g_{\mathcal{X},\mathcal{Y}}^{-1}\left(\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}\right)\right)\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}} \leq \mathcal{E}(u), \quad \forall u \in D(\mathcal{E}),$$
(1.7)

where $\|u\|_{\mathcal{X}} = +\infty$ if $u \notin \mathcal{X}$ and similarly for $\mathcal{Y}$.

b. If $\mathcal{X} = \mathcal{Y}$, and $\psi_{\mathcal{X},\mathcal{Y}}(\lambda) = C_1\lambda^\alpha$, $\alpha > -1$ then the estimate (1.7) reduces to the (Φ, p)-WPI as in Definition 1.3 with $\Phi(u) = \|u\|_{\mathcal{X}}^2$, $p = \frac{\alpha+2}{\alpha+1}$ and $C = C_1^{\frac{1}{2+\alpha}} \frac{2+\alpha}{1+\alpha}$.

Organization of the paper. Before proceeding to prove our theorems we first discuss both the classical and the weak Poincaré inequalities, and their connection to Markov semigroups in Section 2. The proofs will follow in Section 3 and we then present various applications of these theorems in Section 4, where we shall also prove Corollary 1.11.

2. POINCARÉ INEQUALITIES

In this section we recall the famous Poincaré inequality, its connection to Markov semigroups, and we discuss its "weak" variant, the so-called "weak Poincaré inequality."

2.1. **The classical Poincaré inequality.** When M is a compact Riemannian manifold or a bounded domain of $\mathbb{R}^d$, the classical L^2 Poincaré inequality reads [3, §4.2.1]

$$\int_M \left|\varphi(x) - \left(\frac{1}{|M|}\int_M \varphi(y)\,\mathrm{d}y\right)\right|^2 \mathrm{d}x \leq C_M \int_M |\nabla\varphi(x)|^2 \,\mathrm{d}x,$$
(2.1)

where $|M|$ is the volume of M, and $C_M > 0$ is independent of u.

Motivation: the heat semigroup. Let us illustrate why the quantities appearing in this inequality are natural. Let $M \subset \mathbb{R}^d$ be a bounded, connected and smooth domain. Consider the heat semigroup, i.e. solutions of

$$\partial_t u(t, x) = \Delta_x u(t, x), \qquad x \in M, \, t \in \mathbb{R}_+,$$

subject to Neumann boundary conditions with initial data $u(0, x) = u_0(x)$. The associated invariant measure is $\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{|M|}$. It is well-known that in this case the spectrum of Δ_x is discrete and non-positive. In particular, its kernel is separated from the rest of the spectrum.

This immediately implies that $u(t, x) = P_t u_0(x)$ converges to the projection onto the kernel, given by

$$P_{\mathrm{ker}} u_0 := \int_M u_0(x) \, \mathrm{d}\mu(x).$$

Thus, we are interested in the decay rate as $t \to +\infty$ of

$$\mathcal{V}(P_t u_0) := \| P_t u_0 - P_{\mathrm{ker}}(P_t u_0) \|_{L^2(\mathrm{d}\mu)}^2 = \| P_t u_0 - P_{\mathrm{ker}} u_0 \|_{L^2(\mathrm{d}\mu)}^2.$$

The entropy method. A common method to obtain decay rates of this type is the so-called *entropy method.* Given the "relative distance" $\mathcal{V}$ (a Lyapunov functional) we find its *production functional* $\mathcal{E}$ by formally differentiating along the flow of the semigroup:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}(P_t u_0) = 2 \left(\partial_t P_t u_0, P_t u_0 - P_{\mathrm{ker}} u_0 \right)_{L^2(\mathrm{d}\mu)} = 2 \int_M P_t u_0(x) \Delta_x P_t u_0(x) \, \mathrm{d}\mu(x) = -2\mathcal{E}(P_t u_0), \tag{2.2}$$

where $\mathcal{E}$ turns out to be the associated Dirichlet form. Note that since $P_{\mathrm{ker}} = E(\{0\})$ we can rewrite (2.2) as $\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Var}(P_t u_0) = -2\mathcal{E}(P_t u_0)$. Now we seek a pure functional inequality involving $\mathcal{V}$ and $\mathcal{E}$. In particular (see, for example, [17, Chapter 3, §3.2]), one looks for a functional inequality of the form

$$\mathcal{E}(u) \geq \Theta(\mathcal{V}(u)), \qquad \forall u \in D(\mathcal{E}), \tag{2.3}$$

with an appropriate nonnegative function Θ. Succeeding in finding such an inequality entails, in view of (2.2),

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}(P_t u_0) \leq -2\Theta(\mathcal{V}(P_t u_0))$$

from which an explicit rate is derived.

Returning to the heat semigroup, we notice that the classical Poincaré inequality (2.1) is *exactly* a functional inequality of the form of (2.3). Moreover, the linear connection between the variance and the Dirichlet form yields an exponential rate of decay for $\mathrm{Var}(P_t u_0)$.

2.2. **Relationship to Markov semigroups.** In view of Subsection 2.1, there is a natural extension of the notion of a Poincaré inequality to general Markov semigroups. Let $\{P_t\}_{t \geq 0}$ be a Markov semigroup on $\mathcal{H} = L^2(M, \mathrm{d}\mu)$ with a generator $-H$, where H is a self-adjoint, non-negative operator, and $\mathrm{d}\mu$ its invariant measure. Then the Poincaré inequality, as already defined (Definition 1.1), is

$$\mathrm{Var}(u) \leq C\mathcal{E}(u), \qquad \forall u \in D(\mathcal{E}).$$

The following well known theorem (see [3, Theorem 4.2.5]) serves as a motivation for our current investigation:

Theorem 2.1. *The following conditions are equivalent:*
1. *H satisfies a Poincaré inequality with constant C.*
2. *The spectrum of H is contained in $\{0\} \cup \left[\frac{1}{C}, \infty \right)$.*
3. *For every $u \in L^2(M, \mathrm{d}\mu)$ and every $t \geq 0$,*

$$\mathrm{Var}(P_t u) \leq e^{-2t/C} \, \mathrm{Var}(u).$$

2.3. **The weak Poincaré inequality (WPI).** It is natural to ask whether one can obtain a generalization of Theorem 2.1 to generators which lack a spectral gap. We note that a differential operator acting on functions defined in an unbounded domain (generically) lacks a spectral gap. Our Theorems 1.6 and 1.9 provide an answer to this question, where the Poincaré inequality is replaced by some form of a weak Poincaré inequality. In the following we provide a brief review of the existing literature on variants of the weak Poincaré inequality.

This topic has a very rich history, in particular in the second half of the 20th century. As was hinted in Corollary 1.11, a closely related example is Nash's celebrated inequality [15]:

$$\|u\|_{L^2}^2 \leq C \left(\|\nabla u\|_{L^2}^2\right)^{\frac{d}{d+2}} \left(\|u\|_{L^1}^2\right)^{\frac{2}{d+2}}, \qquad \forall u \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$$

where $C > 0$ does not depend on u. Estimates of the same spirit are then developed in [9] for example.

The form of the weak Poincaré inequality which we consider (Definition 1.3) first appeared in [13, Equation (2.3)], where it is also shown how such a differential inequality leads to an algebraic decay rate. These ideas were then further developed in [2, 5, 16, 18–21]. We also refer to [1] where the notion of a "weak spectral gap" is introduced.

In fact, in the influential work of Röckner and Wang [16] several variants of the WPI were introduced. The most general one is

$$\mathrm{Var}(u) \leq \alpha(r)\mathcal{E}(u) + r\Phi(u), \qquad \forall u \in D(\mathcal{E}),\ r > 0,$$

where $\alpha : (0,\infty) \to (0,\infty)$ is decreasing and $\Phi : L^2(\mathrm{d}\mu) \to [0,\infty]$ satisfies $\Phi(cu) = c^2\Phi(u)$ for any $c \in \mathbb{R}$ and $u \in L^2(\mathrm{d}\mu)$. This is equivalent to our (Φ, p)-WPI whenever $\alpha(r) = Cr^{1-p}$.

Continuing upon the work of Röckner and Wang and their notion of WPI, works on connections between these inequalities and isoperimetry or concentration properties of the underlying measures have been extremely prolific in the probability community. We refer the interested reader to [4, 6, 8, 11, 12, 14]. For a recent account of the notions discussed here, and in particular the relationship between functional inequalities and Markov semigroups, we refer to the book [3].

3. Proofs of the theorems

We first prove the more general Theorem 1.9, and show how Theorem 1.6 is a straight-forward corollary. We then show how to obtain the decay rates in Theorem 1.7, and we conclude with the proof of Proposition 1.13. For brevity, we omit the subscripts from the functions $\psi_{\mathcal{X},\mathcal{Y}}$, $\Psi_{\mathcal{X},\mathcal{Y}}$ and $g_{\mathcal{X},\mathcal{Y}}$.

3.1. **Proof of Theorem 1.9a.** First we show that an estimate on the density of states near 0 leads to the WPI (1.6). Let $r_0 \in (0, r)$ to be chosen later. Let $\{E(\lambda)\}_{\lambda \geq 0}$ be the resolution

of the identity of H. Let $u \in D(\mathcal{E}) \cap \mathcal{X} \cap \mathcal{Y}$. Then:

$$\mathcal{E}(u) = \int_M u H u \,\mathrm{d}\mu = \int_M u \int_{[0,\infty)} \lambda \,\mathrm{d}E(\lambda)u \,\mathrm{d}\mu$$

$$\geq \int_M u \int_{[r_0,\infty)} \lambda \,\mathrm{d}E(\lambda)u \,\mathrm{d}\mu \geq r_0 \int_M u \int_{[r_0,\infty)} \mathrm{d}E(\lambda)u \,\mathrm{d}\mu$$

$$= r_0 \int_M u \int_{[0,\infty)} \mathrm{d}E(\lambda)u \,\mathrm{d}\mu - r_0 \|E(\{0\})u\|_{\mathcal{H}}^2 - r_0 \int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda)u \,\mathrm{d}\mu$$

$$= r_0 \operatorname{Var}(u) - r_0 \int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda)u \,\mathrm{d}\mu.$$

We now use the estimate on the density of states (1.5) to obtain

$$\int_M u \int_{(0,r_0)} \mathrm{d}E(\lambda)u \,\mathrm{d}\mu = \int_{(0,r_0)} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(E(\lambda)u, u\right)_{\mathcal{H}} \,\mathrm{d}\lambda$$

$$\leq \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}} \int_{(0,r_0)} \psi(\lambda) \,\mathrm{d}\lambda = \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}\Psi(r_0).$$

Hence we have

$$\mathcal{E}(u) \geq r_0\left(\operatorname{Var}(u) - \|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}\Psi(r_0)\right).$$

Let $K \in (0,1)$ and define

$$r_0 = \Psi^{-1}\left(K\frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}\right) \qquad \text{so that} \qquad \Psi(r_0) = K\frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}}$$

(to satisfy the condition $r_0 < r$ we may need K to be small). Then we get

$$\mathcal{E}(u) \geq r_0(1 - K)\operatorname{Var}(u)$$

which completes the proof.

3.2. **Proof of Theorem 1.9 b & c (and Theorem 1.6).** The proofs follow from the following lemma where we show how (1.6) leads to a (Φ, p)-WPI.

Lemma 3.1. *When $\mathcal{X} = \mathcal{Y}$ and $\psi(\lambda) = C_1\lambda^\alpha$, $\alpha > -1$, the inequality (1.6) reduces to the (Φ, p)-WPI with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ and $p = \frac{\alpha+2}{\alpha+1}$. Furthermore, if $\mathcal{X} = \mathcal{Y} \subset \mathcal{H}$ we recover Theorem 1.6.*

Proof. Let $\psi(\lambda) = C_1\lambda^\alpha$, $\alpha > -1$. Then

$$\Psi(\rho) = C_1 \int_0^\rho \lambda^\alpha \,\mathrm{d}\lambda = \frac{C_1}{\alpha+1}\rho^{\alpha+1}$$

so that

$$\Psi^{-1}(\tau) = \left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}} \tau^{\frac{1}{\alpha+1}}.$$

Hence

$$\Psi^{-1}\left(K\frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right) = \left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}} \left(K\frac{\operatorname{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}}.$$

Plugging this into (1.6) we have

$$\mathcal{E}(u) \geq (1 - K)\Psi^{-1}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)\mathrm{Var}(u)$$

$$= (1 - K)\left(\frac{\alpha + 1}{C_1}\right)^{\frac{1}{\alpha+1}}\left(K\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}\right)^{\frac{1}{\alpha+1}}\mathrm{Var}(u)$$

$$= C'\,\mathrm{Var}(u)^{\frac{\alpha+2}{\alpha+1}}\left(\|u\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}.$$

This leads to

$$\mathrm{Var}(u) \leq C''\mathcal{E}(u)^{\frac{\alpha+1}{\alpha+2}}\left(\|u\|_{\mathcal{X}}^2\right)^{\frac{1}{\alpha+2}}$$

which is a (Φ, p)-WPI with $\Phi(u) = \|u\|_{\mathcal{X}}^2$ and $p = \frac{\alpha+2}{\alpha+1}$. $\qquad\square$

3.3. **Proof of Theorem 1.7.** We show that the growth rate assumption (1.3) leads to a decay of the variance as in (1.4). This proof is rather standard and is included for completeness. Using (2.2), the (Φ, p)-WPI and (1.3), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(P_t u) = -2\mathcal{E}(P_t u) \leq -2C'\,\mathrm{Var}(P_t u)^{\frac{\alpha+2}{\alpha+1}}\left(\|P_t u\|_{\mathcal{X}}^2\right)^{-\frac{1}{\alpha+1}}$$

$$\leq -2C'\,\mathrm{Var}(P_t u)^{\frac{\alpha+2}{\alpha+1}}\left(\|u\|_{\mathcal{X}}^2 + C_2 t^\beta\right)^{-\frac{1}{\alpha+1}}$$

where C' is as in the proof of Lemma 3.1. This is an ordinary differential inequality for $y(t) := \mathrm{Var}(P_t u)$ of the form

$$\dot{y} \leq -Ay^{1+a}(B + Ct^b)^{-c},$$

for $a, c, A, B > 0$, $b \in \mathbb{R}$, and $C \geq 0$. We readily obtain

$$y(t) \leq \left(y(0)^{-a} + aA\int_0^t \left(B + Cs^b\right)^{-c}\mathrm{d}s\right)^{-1/a}$$

which yields the bound (1.4). Asymptotically, we have

$$y(t) = O(t^{-1/a}) \text{ as } t \to +\infty, \qquad \text{if } C = 0 \text{ or } b \leq 0.$$

Otherwise, it is easy to see that $bc = 1$ leads to logarithmic decay, while $bc < 1$ leads to polynomial decay. The precise rates are

$$y(t) = \begin{cases} O((\log t)^{-1/a}) \text{ as } t \to +\infty, & bc = 1. \\ O(t^{-(1-bc)/a}) \text{ as } t \to +\infty, & bc < 1. \end{cases}$$

This completes the proof of Theorem 1.7.

Remark 3.2 (The constant C_3). It is beneficial to provide a detailed computation of the constant C_3 appearing in (1.4). The following computations are performed up to a constant C which does not depend on $\alpha, M, \mathcal{H}, \mathcal{X}$ or any other fundamental quantity.

Considering the proof of Theorem 1.7, we see that C_3 is denoted aA where $a = \frac{1}{\alpha+1}$ and $A = 2C'$ with $C' = \underbrace{(1 - K)K^{\frac{1}{1+\alpha}}}_{\tilde{K}}\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}}$ where C_1 and α appear in the bound (1.2). We readily obtain

$$C_3 = 2\tilde{K}\frac{1}{\alpha+1}\left(\frac{\alpha+1}{C_1}\right)^{\frac{1}{\alpha+1}} = 2\tilde{K}\,(\alpha+1)^{\frac{-\alpha}{\alpha+1}}C_1^{\frac{-1}{\alpha+1}}.$$

In fact, a short computation using the result of Proposition 1.13 yields the even more explicit formula

$$C_3 = 2(\alpha + 2)^{-\frac{\alpha+2}{\alpha+1}} (\alpha + 1)^{\frac{1}{\alpha+1}} C_1^{\frac{-1}{\alpha+1}}.$$

3.4. **Proof of Proposition 1.13.** As seen in the Proof of Theorem 1.6 we have that for all $r_0 \in (0, r)$

$$\mathcal{E}(u) \geq r_0 \left(\mathrm{Var}(u) - \|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}} \Psi(r_0) \right). \tag{3.1}$$

Our goal is to maximize the right hand side of this inequality. As such, for any $a, b > 0$, consider the function

$$h(\rho) = \rho \left(a - \Psi(\rho) b \right).$$

By assumption, we can extend ψ to a continuous function on $(0, R)$, so that h is differentiable and we have

$$h'(\rho) = a - g(\rho) b.$$

As g increases from 0 to $+\infty$ we see that the unique critical point, $\rho = g^{-1}\left(\frac{a}{b}\right)$ is a maximum point of h. Thus

$$\max_{\rho \in (0, R)} h(\rho) = g^{-1}\left(\frac{a}{b}\right) \left(a - \Psi\left(g^{-1}\left(\frac{a}{b}\right) \right) b \right)$$

$$= g^{-1}\left(\frac{a}{b}\right) \left(a - \left[g\left(g^{-1}\left(\frac{a}{b}\right) \right) - g^{-1}\left(\frac{a}{b}\right) \psi\left(g^{-1}\left(\frac{a}{b}\right) \right) \right] b \right)$$

$$= g^{-1}\left(\frac{a}{b}\right)^2 \psi\left(g^{-1}\left(\frac{a}{b}\right) \right) b.$$

Applying this maximization process to the right hand side of (3.1) with $a = \mathrm{Var}(u)$ and $b = \|u\|_{\mathcal{X}} \|u\|_{\mathcal{Y}}$ yields the desired inequality (1.7).

To show the second part of the theorem we notice that $\psi(\lambda)$ can be extended to a continuous function on $(0, +\infty)$ with the same formula $C_1 \lambda^\alpha$. The expression for Ψ is $\Psi(\rho) = C_1 \frac{\rho^{1+\alpha}}{1+\alpha}$. We note that

$$g(\rho) = C_1 \frac{2 + \alpha}{1 + \alpha} \rho^{1+\alpha}$$

satisfies the conditions $\lim_{\rho \to 0} g(\rho) = 0$ and $\lim_{\rho \to +\infty} g(\rho) = +\infty$. Since

$$g^{-1}(y) = \left(\frac{1 + \alpha}{C_1 (2 + \alpha)} \right)^{\frac{1}{1+\alpha}} y^{\frac{1}{1+\alpha}}$$

and

$$g^{-1}(y)^2 \psi\left(g^{-1}(y) \right) = C_1 \left(g^{-1}(y) \right)^{\alpha+2}$$

we obtain the result by substituting $y = \frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2}$ thus leading to the inequality

$$\mathcal{E}(u) \geq C_1^{-\frac{1}{1+\alpha}} \left(\frac{1 + \alpha}{2 + \alpha} \right)^{\frac{2+\alpha}{1+\alpha}} \left(\frac{\mathrm{Var}(u)}{\|u\|_{\mathcal{X}}^2} \right)^{\frac{2+\alpha}{1+\alpha}} \|u\|_{\mathcal{X}}^2.$$

4. EXAMPLES

Here we consider several notable examples of equations

$$\begin{cases} \partial_t u(t,x) = -Hu(t,x), & t \in \mathbb{R}_+, \, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where H is a **constant coefficient pseudodifferential operator**:

$$H = P(D).$$

With a slight abuse of notation, we write $H = P(\xi)$, where $\xi \in \mathbb{R}^d$.

Assumption A3. Assume that there exist $\gamma_1 > -1$ and $C, \gamma_2 > 0$ so that $P(\xi)$ satisfies the following conditions:

(1) $P(0) = 0$,
(2) $C^{-1}|\xi|^{\gamma_1+1} \leq P(\xi) \leq C|\xi|^{\gamma_2}$, for any $\xi \in \mathbb{R}^d$,
(3) $C^{-1}|\xi|^{\gamma_1} \leq |\nabla P(\xi)|$, for any $\xi \in \mathbb{R}^d \setminus \{0\}$,
(4) $\mathcal{H}^{d-1}\left(\{\xi \in \mathbb{R}^d : P(\xi) = \lambda\}\right) \leq C\lambda^{\frac{d-1}{\gamma_1+1}}$, for any $\lambda > 0$.

Here $\mathcal{H}^{d-1}$ is the $d-1$-dimensional Hausdorff measure. (We use the same constant C in all inequalities for simplicity, but one could specify different constants)

Then since $P(\xi)$ is a multiplication operator, one obtains the following simple expression for the spectral measure $E(\lambda)$ of H:

$$(E(\lambda)u, v)_{L^2} = \int_{P(\xi) \leq \lambda} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi. \tag{4.1}$$

Let $d\sigma_{\lambda_0}$ denote the uniform Lebesgue measure on the surface $\{\xi \in \mathbb{R}^d : P(\xi) = \lambda_0\}$. Then differentiating (4.1) and using the coarea formula we obtain:

$$\left| \frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} (E(\lambda)u, v)_{L^2(\mathbb{R}^d)} \right| = \left| \int_{P(\xi)=\lambda_0} \frac{1}{|\nabla P(\xi)|} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\sigma_{\lambda_0} \right|$$

$$\leq \left| \int_{P(\xi)=\lambda_0} \frac{1}{|\nabla P(\xi)|} \, d\sigma_{\lambda_0} \right| \|\widehat{u}\|_{L^\infty(\mathbb{R}^d)} \|\widehat{v}\|_{L^\infty(\mathbb{R}^d)}$$

$$\leq \left| \int_{P(\xi)=\lambda_0} \frac{1}{|\nabla P(\xi)|} \, d\sigma_{\lambda_0} \right| \|u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^1(\mathbb{R}^d)}$$

$$\leq C \int_{P(\xi)=\lambda_0} |\xi|^{-\gamma_1} \, d\sigma_{\lambda_0} \|u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^1(\mathbb{R}^d)}$$

$$\leq C \int_{P(\xi)=\lambda_0} \lambda_0^{-\gamma_1/\gamma_2} \, d\sigma_{\lambda_0} \|u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^1(\mathbb{R}^d)}$$

$$\leq C\lambda_0^{-\frac{\gamma_1}{\gamma_2}+\frac{d-1}{\gamma_1+1}} \|u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^1(\mathbb{R}^d)}. \tag{4.2}$$

From Theorem 1.9b this bound on the DoS leads to a (Φ, p)-WPI with $\mathcal{X} = L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\Phi(u) = \|u\|_{L^1}^2$ and $p = \frac{2 - \frac{\gamma_1}{\gamma_2} + \frac{d-1}{\gamma_1+1}}{1 - \frac{\gamma_1}{\gamma_2} + \frac{d-1}{\gamma_1+1}} = 1 + \frac{\gamma_2(\gamma_1+1)}{\gamma_2\gamma_1 - \gamma_1(\gamma_1+1) + d\gamma_2}$:

$$\|u\|_{L^2}^2 \leq C\mathcal{E}(u)^{\frac{\gamma_2\gamma_1 - \gamma_1(\gamma_1+1) + d\gamma_2}{2\gamma_2(\gamma_1+\frac{1}{2}) - \gamma_1(\gamma_1+1) + d\gamma_2}} \|u\|_{L^1}^{2 - \frac{2\gamma_2\gamma_1 - 2\gamma_1(\gamma_1+1) + 2d\gamma_2}{2\gamma_2(\gamma_1+\frac{1}{2}) - \gamma_1(\gamma_1+1) + d\gamma_2}}. \tag{4.3}$$

Moreover, if $\gamma_2 = 1 + \gamma_1$ the power of λ_0 in the bound (4.2) simply becomes $\lambda_0^{\frac{d}{\gamma_2}-1}$ and then (4.3) simplifies to

$$\|u\|_{L^2}^2 \le C\mathcal{E}(u)^{\frac{d}{\gamma_2+d}}\|u\|_{L^1}^{\frac{2\gamma_2}{\gamma_2+d}}. \tag{4.4}$$

Remark 4.1 (Other functional subspaces). We focus here on solutions lying in L^1. However, other natural subspaces to consider are the Hilbert subspaces $L^{2,s}(\mathbb{R}^d)$, defined as

$$L^{2,s}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \ : \ \|u\|_{L^{2,s}(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} |u(x)|^2 (1+|x|^2)^s \, \mathrm{d}x < \infty \right\}.$$

These are naturally obtained as follows. In the estimate (4.2) above, rather than extract $\widehat{u}$ and $\widehat{v}$ in L^∞, one can use the trace lemma to estimate them in H^s with $s > 1/2$ (if the surface is sufficiently regular for the trace lemma to hold). Then, one uses the simple observation that the $L^{2,s}$ norm of a function is the same as the H^s norm of its Fourier transform. The main difference is that the power of λ_0 in the resulting inequality will be different.

4.1. **The Laplacian.** For the Laplacian $P(\xi) = |\xi|^2$, the associated equation is the heat equation:

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x), & t \in \mathbb{R}_+, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Assumption A3 is satisfied with $\gamma_2 = \gamma_1 + 1 = 2$, so that the DoS is estimated by $\lambda_0^{\frac{d}{2}-1}$:

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_0} (E(\lambda)u,v)_{L^2} \right| \le C\lambda_0^{\frac{d}{2}-1}\|u\|_{L^1(\mathbb{R}^d)}\|v\|_{L^1(\mathbb{R}^d)}.$$

Then the WPI (4.4) becomes

$$\|u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^{\frac{2d}{2+d}}\|u\|_{L^1}^{\frac{4}{2+d}}. \tag{4.5}$$

Remark 4.2 (Nash inequality). This functional inequality is precisely the Nash inequality. This demonstrates how our methodology gives a general framework for many known important inequalities, presented in general form in (4.3) and (4.4).

Remark 4.3 (The constant in the Nash inequality). We note that the computation (4.2) can be performed with precise constants in this case. Then, using Proposition 1.13, we may extract a precise constant in (4.5). A simple computation yields the constant $C = \left(\frac{|\mathbb{S}^{d-1}|}{2}\right)^{\frac{2}{2+d}}\frac{2+d}{d}$. These computations are left to the reader. We note that the optimal constant in the Nash inequality has already been obtained long ago by Carlen and Loss [10]. Improving our constant is the subject of ongoing research.

Convergence to equilibrium. We can apply Theorems 1.9c and 1.7 with $\alpha = \frac{d}{2}-1$ and $\Phi(u) = \|u\|_{L^1}^2$. Using the fact that the L^1 norm of solutions to the heat equation do not increase we have $C_2 = 0$, where C_2 is the constant appearing in (1.3). The bound (1.4)

becomes

$$\mathrm{Var}(u(t,\cdot)) \le \left(\mathrm{Var}(u_0)^{\frac{-1}{1+\alpha}} + C \int_0^t \|u_0\|_{L^1}^{\frac{-2}{1+\alpha}} \, \mathrm{d}s \right)^{-(1+\alpha)}$$

$$= \left(\mathrm{Var}(u_0)^{-\frac{2}{d}} + C\|u_0\|_{L^1}^{-\frac{4}{d}} t \right)^{-\frac{d}{2}}$$

$$\le C\|u_0\|_{L^1}^2 t^{-\frac{d}{2}}$$

and we conclude that for every $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$$\|u(t,\cdot)\|_{L^2}^2 = \mathrm{Var}(u(t,\cdot)) = O(t^{-\frac{d}{2}}), \qquad \text{as } t \to +\infty,$$

which is the optimal rate for the heat equation. This can be extended to any $u_0 \in L^1(\mathbb{R}^d)$ by density.

4.2. **The fractional Laplacian.** For $P(\xi) = |\xi|^{2p}$ ($p \in (0,1)$) Assumption A3 on $P(\xi)$ is satisfied with $\gamma_2 = \gamma_1 + 1 = 2p$, so that the DoS is estimated by $\lambda_0^{\frac{d}{2p}-1}$ and (4.4) becomes

$$\|u\|_{L^2}^2 \le C\|(-\Delta)^{\frac{p}{2}}u\|_{L^2}^{\frac{2d}{2p+d}} \|u\|_{L^1}^{\frac{4p}{2p+d}}.$$

Remark 4.4. There is no reason not to take values of p greater than 1. However, the restriction to $p \in (0,1)$ is quite common in the literature, and the result below on time decay only applies to $p \in (0,1)$.

Convergence to equilibrium. From [7] we know that $\|u(t,\cdot)\|_{L^1} \le \|u_0\|_{L^1}$ and as such, much like the previous example, we conclude that

$$\mathrm{Var}(u(t,\cdot)) \le \left(\mathrm{Var}(u_0)^{-\frac{2p}{d}} + C\|u_0\|_{L^1}^{-\frac{4p}{d}} t \right)^{-\frac{d}{2p}} \le C\|u_0\|_{L^1}^2 t^{-\frac{d}{2p}}$$

and hence the asymptotic decay rate is

$$\|u(t,\cdot)\|_{L^2}^2 = \mathrm{Var}(u(t,\cdot)) = O(t^{-\frac{d}{2p}}), \qquad \text{as } t \to +\infty.$$

4.3. **Homogeneous elliptic operators.** Consider homogenous elliptic operators of the form

$$P(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \qquad m \in \{2,4,6,\dots\},$$

where $\alpha \in \mathbb{N}_0^d$ is a multi-index with $|\alpha| = \sum_{i=1}^d \alpha_i$ and where all coefficients $a_\alpha \in \mathbb{R}$ are *assumed* to be such that the operator satisfies Assumption A3. In this case $m = \gamma_1 + 1 = \gamma_2$ and the WPI (4.4) becomes

$$\|u\|_{L^2}^2 \le C\|P^{1/2}(D)u\|_{L^2}^{\frac{2d}{m+d}} \|u\|_{L^1}^{\frac{2m}{m+d}}.$$

Examples of such operators which are not functions of the Laplacian include:

(1) $P(\xi) = \sum_{i=1}^d |\xi_i|^4$,
(2) $P(\xi) = \sum_{i=1}^d |\xi_i|^2 - \xi_1 \xi_2$.

For these examples, the only nontrivial condition to verify is the condition $\mathcal{H}^{d-1}\left(\{\xi \in \mathbb{R}^d : P(\xi) = \lambda\}\right) \le C\lambda^{\frac{d-1}{m}}$.

Convergence to equilibrium. In order to prove convergence to an equilibrium state, one has to know how the L^1 norm behaves under the flow. The authors are not aware of

results in the literature for general operators as the ones we consider here. Based on the known results for the Laplacian and the fractional Laplacian one could ask:

Question 4.5. *Is it true that for every homogeneous elliptic operator of order m which satisfies Assumption A3 and which is the generator of a semigroup $(P_t)_{t \geq 0}$ there exist $C_2 = C_2(u) \geq 0$ and $\beta \in \mathbb{R}$ such that for every $t \geq 0$, $\|P_t u\|_{L^1}^2 \leq \|u\|_{L^1}^2 + C_2 t^{\beta}$?*

If the answer is 'yes', from Theorem 1.7 this conjecture leads to the following rate of convergence to equilibrium:

$$\mathrm{Var}(P_t u) \leq \begin{cases} O((\log t)^{-\frac{d}{m}}) & \beta = \frac{d}{m}. \\ O(t^{\beta - \frac{d}{m}}) & 0 < \beta < \frac{d}{m}. \\ O(t^{-\frac{d}{m}}) & C_2 = 0 \text{ or } \beta \leq 0. \end{cases}$$

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