Approximations of strongly continuous families of unbounded self-adjoint operators

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Abstract

The problem of approximating the discrete spectra of families of self-adjoint operators that are merely strongly continuous is addressed. It is well-known that the spectrum need not vary continuously (as a set) under strong perturbations. However, it is shown that under an additional compactness assumption the spectrum does vary continuously, and a family of symmetric finite-dimensional approximations is constructed. An important feature of these approximations is that they are valid for the entire family simultaneously. An application of this result to the study of plasma instabilities is illustrated.

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1 Introduction

1.1 Overview

We present a method for obtaining finite-dimensional approximations of the discrete spectrum of families of self-adjoint operators. We are interested in operators that decompose into a system of two coupled Schrödinger operators with opposite signs (see (1.1) below). However our results are applicable to "standard" Schrödinger operators, and in fact we prove our main result, Theorem 2, for Schrödinger operators first, see Theorem 2′. We are interested in the following problem:

Problem 1. Consider the family of self-adjoint unbounded operators

$$\mathcal{M}^\lambda = \mathcal{A} + \mathcal{K}^\lambda = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1] \tag{1.1}$$

acting in an appropriate subspace of $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, where $\{\mathcal{K}^\lambda\}_{\lambda\in[0,1]}$ is a bounded, symmetric and strongly continuous family. *Is it possible to construct explicit finite-dimensional symmetric approximations of $\mathcal{M}^\lambda$ whose spectrum in $(-1,1)$ converges to that of $\mathcal{M}^\lambda$ for all λ simultaneously?*

This problem is motivated by Maxwell's equations, which in the Lorentz gauge may be written as the following elliptic system for the electromagnetic potentials ϕ and $\mathbf{A}$ (after taking a Laplace transform in time):

$$\begin{cases} (-\boldsymbol{\Delta} + \lambda^2)\mathbf{A} + \mathbf{j} = \mathbf{0} \\ (\Delta - \lambda^2)\phi + \rho = 0 \end{cases} \tag{1.2}$$

where ρ and $\mathbf{j}$ are the charge and current densities, respectively. The specific problem we have in mind, treated separately in [2], is that of instabilities of the relativistic Vlasov-Maxwell system describing the evolution of collisionless plasmas; it is outlined in section 6 below. The Vlasov equation provides the coupling of the two equations in (1.2), making the system self-adjoint (see, for instance, the expressions (6.5) and (6.6)).

1.2 The main result

Let $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ be a (separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let

$$\mathcal{A}^\lambda = \begin{bmatrix} \mathcal{A}_+^\lambda & 0 \\ 0 & -\mathcal{A}_-^\lambda \end{bmatrix} \quad \text{and} \quad \mathcal{K}^\lambda = \begin{bmatrix} \mathcal{K}_{++}^\lambda & \mathcal{K}_{+-}^\lambda \\ \mathcal{K}_{-+}^\lambda & \mathcal{K}_{--}^\lambda \end{bmatrix}, \quad \lambda \in [0,1]$$

be two families of operators on $\mathfrak{H}$ depending upon the parameter $\lambda \in [0,1]$, where the family $\mathcal{A}^\lambda$ is also assumed to be defined for λ in an open neighbourhood D_0 of $[0,1]$ in the complex plane. They satisfy:

i) Sectoriality: The sesquilinear forms $\mathfrak{a}_\pm^\lambda$ corresponding to $\mathcal{A}_\pm^\lambda$ are sectorial for $\lambda \in D_0$, symmetric for real λ, have dense domains $\mathfrak{D}(\mathfrak{a}_\pm^\lambda)$ independent of $\lambda \in D_0$,[1] and $D_0 \ni \lambda \mapsto \mathfrak{a}_\pm^\lambda[u,v]$ are holomorphic for any $u, v \in \mathfrak{D}(\mathfrak{a}_\pm^\lambda)$. [In the terminology of Kato [5], $\mathfrak{a}_\pm^\lambda$ are *holomorphic families of type (a)* and $\mathcal{A}^\lambda$ are *holomorphic families of type (B).*]

ii) Gap: $\mathcal{A}_\pm^\lambda > 1$ for every $\lambda \in [0,1]$.

iii) Bounded perturbation: $\{\mathcal{K}^\lambda\}_{\lambda\in[0,1]} \subset \mathfrak{B}(\mathfrak{H})$ is a symmetric strongly continuous family.

iv) Compactness: There exist symmetric operators $\mathcal{P}_\pm \in \mathfrak{B}(\mathfrak{H}_\pm)$ which are relatively compact with respect to the forms $\mathfrak{a}_\pm^\lambda$, satisfying $\mathcal{K}^\lambda = \mathcal{K}^\lambda\mathcal{P}$ for all $\lambda \in [0,1]$ where

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{bmatrix}.$$

[1] Hence we shall henceforth remove the λ superscript when discussing the domains of $\mathfrak{a}^\lambda$ and $\mathfrak{a}_\pm^\lambda$.

Finally, if the family $\mathcal{A}^\lambda$ does not have a compact resolvent we assume:

v) Compactification of the resolvent: There exist holomorphic forms $\{\mathfrak{w}_\pm^\lambda\}_{\lambda\in D_0}$ of type (a) and associated operators $\{\mathcal{W}_\pm^\lambda\}_{\lambda\in D_0}$ of type (B) such that for $\lambda\in[0,1]$, $\mathcal{W}_\pm^\lambda$ are self-adjoint and non-negative, and if $\mathfrak{w}^\lambda$ is the form associated with

$$\mathcal{W}^\lambda = \begin{bmatrix} \mathcal{W}_+^\lambda & 0 \\ 0 & -\mathcal{W}_-^\lambda \end{bmatrix}, \quad \lambda \in D_0,$$

then $\mathfrak{D}(\mathfrak{w}^\lambda)\cap\mathfrak{D}(\mathfrak{a}_\pm)$ are dense for all $\lambda\in D_0$ and the inclusion $(\mathfrak{D}(\mathfrak{w}^\lambda)\cap\mathfrak{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}_\varepsilon^\lambda}) \to (\mathfrak{H}, \|\cdot\|)$ is compact for some $\lambda\in D_0$ and all $\varepsilon>0$, where $\mathfrak{a}_\varepsilon^\lambda$ is the form associated with

$$\mathcal{A}_\varepsilon^\lambda := \mathcal{A}^\lambda + \varepsilon\mathcal{W}^\lambda, \quad \lambda\in D_0,\ \varepsilon\ge 0. \tag{1.3}$$

We can now define the family of (unbounded) operators $\{\mathcal{M}^\lambda\}_{\lambda\in[0,1]}$, acting in $\mathfrak{H}$, as

$$\mathcal{M}^\lambda = \mathcal{A}^\lambda + \mathcal{K}^\lambda, \quad \lambda\in[0,1]. \tag{1.4}$$

Our main result is formulated with the general case of continuous spectrum in mind:

Theorem 2. *Let $\mathcal{A}_\varepsilon^\lambda$ be as in (1.3), and define*

$$\mathcal{M}_\varepsilon^\lambda = \mathcal{A}_\varepsilon^\lambda + \mathcal{K}^\lambda, \quad \lambda\in[0,1]. \tag{1.5}$$

Let $\{e_{\varepsilon,k}^\lambda\}_{k\in\mathbb{N}}\subset\mathfrak{H}$ be a complete orthonormal set of eigenfunctions of $\mathcal{A}_\varepsilon^\lambda$, let $\mathcal{G}_{\varepsilon,n}^\lambda:\mathfrak{H}\to\mathfrak{H}$ be the orthogonal projection operators onto $\operatorname{span}(e_{\varepsilon,1}^\lambda,\dots,e_{\varepsilon,n}^\lambda)$ *and let $\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda$ be the n-dimensional operator defined as the restriction of $\mathcal{M}_\varepsilon^\lambda$ to $\mathcal{G}_{\varepsilon,n}^\lambda(\mathfrak{H})$. Fix $\varepsilon^*>0$, and define the function*

$$\Sigma : [0,1]\times[0,\varepsilon^*] \to (\textit{subsets of } (-1,1), d_H)$$
$$\Sigma(\lambda,\varepsilon) = (-1,1)\cap\operatorname{sp}(\mathcal{M}_\varepsilon^\lambda)$$

and for fixed $\varepsilon>0$ the function

$$\Sigma_\varepsilon : [0,1]\times\overline{\mathbb{N}} \to (\textit{subsets of } (-1,1), d_H)$$
$$\Sigma_\varepsilon(\lambda,n) = (-1,1)\cap\operatorname{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda)$$

where d_H is the Hausdorff distance (defined below), $\overline{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$, $\operatorname{sp}(\mathcal{M})$ is the spectrum of $\mathcal{M}$, and where we use the convention that $\widetilde{\mathcal{M}}_{\varepsilon,\infty}^\lambda := \mathcal{M}_\varepsilon^\lambda$. Then Σ and Σ_ε are continuous functions of their arguments in the standard topologies on $\mathbb{R}$ (and its subsets) and $\overline{\mathbb{N}}$.

We recall the definition of the Hausdorff distance between two bounded sets $X, Y \subset \mathbb{C}$:

$$d_H(X,Y) = \max\left(\sup_{y\in Y}\inf_{x\in X}|x-y|, \sup_{x\in X}\inf_{y\in Y}|x-y|\right).$$

Theorem 2 provides the following answer to Problem 1: it is indeed possible to approximate the family of operators simultaneously. This is achieved by two levels of approximations: first *ε-approximations* that discretise the spectrum, then *n-approximations* that truncate the problem.

As the notation becomes quite cumbersome due to the decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, we shall first treat the simpler case of semi-bounded operators. So as to avoid repetitions in presentation, we think of the semi-bounded case as the same as before, with $\mathfrak{H} = \mathfrak{H}_+$ and the subspace $\mathfrak{H}_-$ being trivial. For brevity we drop the $+$ subscript. The proof of Theorem 2 is presented in section 5, after the following theorem is proved:

Theorem 2′. *In the case $\mathfrak{H} = \mathfrak{H}_+$ the same conclusion of Theorem 2 holds with* $\mathrm{Ran}(\Sigma) = \mathrm{Ran}(\Sigma_\varepsilon) = ($*bounded subsets of* $(-\infty,1), d_H)$ *defined as*

$$\Sigma(\lambda,\varepsilon) = (-\infty,1)\cap \mathrm{sp}(\mathcal{M}_\varepsilon^\lambda)$$

and

$$\Sigma_\varepsilon(\lambda,n) = (-\infty,1)\cap \mathrm{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda).$$

Remark 3. We note that in both theorems, by the Heine-Cantor theorem, the two maps Σ and Σ_ε are in fact uniformly continuous.

1.3 Discussion

One of the main driving forces behind the study of linear operators in the 20th century was the development of quantum mechanics. Particular attention had been given to the characterisation of the spectra of such operators, as it encodes many important physical properties (such as energy levels, for instance). When operators become too complex, a typical approach is to view them as perturbations of simpler operators whose spectrum is well understood. Two of the classic texts on this topic are those written by Kato [5] and Reed and Simon [9]. Both are still widely cited to this day. We also refer to Simon's review paper [10] and the references therein.

Recently, Hansen [4] presented new techniques for approximating spectra of linear operators (self-adjoint and non-self-adjoint) from a more computational point of view. In [11], Strauss presents a new method for approximating eigenvalues and eigenvectors of self-adjoint operators via an algorithm that is itself self-adjoint, and which does not produce spectral pollution. Both papers provide extensive references to additional literature in the field. We also mention [6], where analysis similar to ours is performed for bounded operators. We note that spectral pollution (the appearance of spurious eigenvalues within gaps in the essential spectrum when approximating) has attracted significant attention [3, 7, 8]. The gap that we have in the spectrum is of a different nature (it is due to the way the problem decomposes), and therefore pollution is less of a concern.

The question that we are motivated by is somewhat different. We are interested in the simultaneous approximation of *families of operators*, rather than approximating a single fixed linear operator. This may be viewed as perturbation theory with two parameters: the continuous parameter λ representing small continuous perturbations generating the family of operators, and the discrete parameter n representing the dimension of the finite-dimensional approximation. One of the important aspects of this theory is that the finite-dimensional approximations apply to the entire family of operators simultaneously. Previously, in [1, Proposition 2.5] a much weaker result of this type was obtained, where the resolvent set of Schrödinger operators with a compact resolvent was shown to be stable under similar perturbations.

There are two substantial difficulties in proving these theorems. If the spectrum of $\mathcal{A}^\lambda$ were discrete for some λ (and therefore for all λ) we would have a natural way to construct approximations by projecting onto increasing subspaces associated to the eigenvalues of $\mathcal{M}^\lambda$. However we do not require the spectrum to be discrete, and, indeed, in the type of problems we have in mind it is not. This necessitates the introduction of yet another perturbation parameter, ε, related to the compactification of the resolvent. The other difficulty is in ensuring that the finite-dimensional approximations approximate the whole family of operators simultaneously. To this end, the compactness assumption (iv) plays a crucial role (see Remark 9 below).

We make several remarks on Theorem 2 and Theorem 2$'$ and the assumptions (i)-(v):

Remark 4. The compactness requirements (iv) on $\mathcal{P}$ are motivated by (1.1). If $\mathcal{A}$ has a compact resolvent (e.g. when acting in $L^2(\mathbb{T}^d) \oplus L^2(\mathbb{T}^d)$ where $\mathbb{T}^d$ is the d-dimensional torus) we may take $\mathcal{P}$ to be the identity. Otherwise (e.g. for $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$) if the perturbations $\mathcal{K}^\lambda$ are compactly supported in the sense that

$$\bigcup_{\lambda \in [0,1], u \in \mathfrak{H}} \operatorname{supp}(\mathcal{K}^\lambda u) \subset K \tag{1.6}$$

where $K = K_+ \times K_- \subset \mathbb{R}^d \times \mathbb{R}^d$ is compact, then we may take $\mathcal{P}_\pm$ as multiplications by the indicator functions of the sets $K_\pm$. Indeed, we first note that (1.6) implies that for all λ, $\mathcal{K}^\lambda = \mathcal{P}\mathcal{K}^\lambda$. Then as $\mathcal{K}^\lambda$ and $\mathcal{P}$ are symmetric, we deduce that $\mathcal{K}^\lambda = (\mathcal{K}^\lambda)^* = (\mathcal{K}^\lambda)^*\mathcal{P}^* = \mathcal{K}^\lambda\mathcal{P}$ as required. That $\mathcal{P}$ is relatively compact with respect to $-\Delta$ follows from Rellich's theorem. We also remark that this choice of $\mathcal{P}$ is in fact the natural inclusion map from L^2 to $L^2(K)$.

Remark 5. Care must be taken regarding the spaces we view operators as acting on. If we view $\mathcal{M}^\lambda_{\varepsilon,n} = \mathcal{G}^\lambda_{\varepsilon,n}\mathcal{M}^\lambda_\varepsilon\mathcal{G}^\lambda_{\varepsilon,n} : \mathfrak{H} \to \mathfrak{H}$ then 0 will *always* be a spurious eigenvalue with infinite multiplicity. To remove this unwanted eigenvalue we must instead consider $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n} : \mathfrak{H}^\lambda_{\varepsilon,n} \to \mathfrak{H}^\lambda_{\varepsilon,n}$ where $\mathfrak{H}^\lambda_{\varepsilon,n} = \mathcal{G}^\lambda_{\varepsilon,n}(\mathfrak{H})$ is the n-dimensional space corresponding to the eigenprojection $\mathcal{G}^\lambda_{\varepsilon,n}$.

Remark 6. Property (ii) implies that there exists $\alpha(\lambda) > 0$ such that $(-\alpha(\lambda) - 1, 1 + \alpha(\lambda))$ is in the resolvent set of $\mathcal{A}^\lambda$. Since the spectrum is continuous in $\lambda \in [0, 1]$ this implies that there is a uniform constant $\alpha > 0$ such that $(-\alpha - 1, 1 + \alpha)$ is in the resolvent set of $\mathcal{A}^\lambda$ for all $\lambda \in [0, 1]$.

Let us summarise some of the notation we use throughout this article. For operators we use upper case calligraphic letter, such as $\mathcal{T}$. As already exhibited above, the spectrum of $\mathcal{T}$ is denoted $\mathrm{sp}(\mathcal{T})$. For the sesquilinear form associated to an operator we use the same letter in lower case Fraktur font. Hence the operator $\mathcal{T}$ has the associated form $\mathfrak{t}$. The space of bounded linear operators on a Hilbert space $\mathfrak{H}$ is denoted $\mathfrak{B}(\mathfrak{H})$. Domains of operators or forms are denoted by $\mathfrak{D}$. The graph norms of an operator $\mathcal{T}$ and a form $\mathfrak{t}$ are denoted $\|\cdot\|_\mathcal{T}$ and $\|\cdot\|_\mathfrak{t}$, respectively. Strong, strong resolvent and norm resolvent convergence are denoted by $\xrightarrow{s}$, $\xrightarrow{s.r.}$ and $\xrightarrow{n.r.}$, respectively. For brevity, we denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

This paper is organised as follows. In section 2 we present some results related to general properties (such as self-adjointness, equivalence of norms, etc.) of the various operators. In section 3 we construct the finite-dimensional approximations to our family of operators, which are used in section 4 to prove Theorem $2'$. In section 5 these results are extended to families of operators which are not positive, proving Theorem 2. Finally, in section 6 we give a brief description of an application of these results, related to plasma instabilities.

2 Preliminary results

We remind the reader that in this section, as well as in section 3 and section 4 we treat the semi-bounded case where $\mathfrak{H} = \mathfrak{H}_+$ and we drop the $+$

subscript.

Considering the definition (1.4) and the subsequent specifications of the properties of the various operators and associated forms, we have the following results.

Lemma 7. *For any $\lambda \in [0,1]$, $\mathcal{M}^\lambda$ is self-adjoint and has the same essential spectrum and domain as $\mathcal{A}^\lambda$. In particular its spectrum inside $(-\infty, 1]$ is discrete. Furthermore, the associated form $\mathfrak{m}^\lambda$ has the same domain as $\mathfrak{a}^\lambda$, which is independent of λ.*

Proof. Self-adjointness follows from the Kato-Rellich theorem, due to $\mathcal{A}^\lambda$ being self-adjoint for $\lambda \in [0,1]$ and the symmetry assumption (iii) on $\mathcal{K}^\lambda$. The essential spectrum result follows from Weyl's theorem as $\mathcal{K}^\lambda = \mathcal{K}^\lambda \mathcal{P}$ is relatively compact with respect to $\mathcal{A}^\lambda$ (for any λ) because $\mathcal{P}$ is. The equality $\mathfrak{D}(\mathfrak{m}^\lambda) = \mathfrak{D}(\mathfrak{a}^\lambda)$ holds since $\mathcal{K}^\lambda$ is bounded for each λ. The fact that the domains are independent of λ was assumed above in the Sectoriality assumption (i). $\square$

Next, we turn our attention to the map $\lambda \mapsto \mathcal{M}^\lambda$. Intuitively, one would expect $\mathcal{M}^\lambda$ to have continuity properties similar to those of $\mathcal{K}^\lambda$ and therefore be merely continuous in the strong resolvent sense. In fact, due to the relative compactness assumption on $\mathcal{P}$ we have more:

Proposition 8. *The family $\{\mathcal{M}^\lambda\}_{\lambda \in [0,1]}$ is norm resolvent continuous.*

Proof. Fix some $\lambda \in [0,1]$ and let $[0,1] \ni \lambda_n \to \lambda$ as $n \to \infty$. It is sufficient to prove

$$\left\| (\mathcal{M}^{\lambda_n} + i)^{-1} - (\mathcal{M}^\lambda + i)^{-1} \right\|_{\mathfrak{B}(\mathfrak{H})} \to 0 \text{ as } n \to \infty.$$

Using the triangle inequality we have

$$\begin{aligned} \left\| (\mathcal{M}^{\lambda_n} + i)^{-1} - (\mathcal{M}^\lambda + i)^{-1} \right\|_{\mathfrak{B}(\mathfrak{H})} &\le \left\| (\mathcal{M}^{\lambda_n} + i)^{-1} - (\mathcal{A}^{\lambda_n} + \mathcal{K}^\lambda + i)^{-1} \right\|_{\mathfrak{B}(\mathfrak{H})} \\ &\quad + \left\| (\mathcal{A}^{\lambda_n} + \mathcal{K}^\lambda + i)^{-1} - (\mathcal{M}^\lambda + i)^{-1} \right\|_{\mathfrak{B}(\mathfrak{H})}. \end{aligned}$$

By observing that $\{\mathcal{A}^\sigma + \mathcal{K}^\lambda\}_{\sigma \in D_0}$ is also a holomorphic family we deduce that the second term tends to zero as $n \to \infty$. For the first term we follow the method used to deduce the second Neumman series (see [5, II-(1.13)])

$$(\mathcal{A}^{\lambda_n} + \mathcal{K}^{\lambda_n} + i)^{-1} = (\mathcal{A}^{\lambda_n} + \mathcal{K}^\lambda + i)^{-1}(1 + (\mathcal{K}^{\lambda_n} - \mathcal{K}^\lambda)(\mathcal{A}^{\lambda_n} + \mathcal{K}^\lambda + i)^{-1})^{-1}$$

which is valid whenever $\left\| (\mathcal{K}^{\lambda_n} - \mathcal{K}^\lambda)(\mathcal{A}^{\lambda_n} + \mathcal{K}^\lambda + i)^{-1} \right\|_{\mathfrak{B}(\mathfrak{H})} < 1$. By the norm resolvent continuity of operator inversion and again using the norm resolvent continuity of the family $\{\mathcal{A}^\sigma + \mathcal{K}^\lambda\}_{\sigma\in[0,1]}$, it is sufficient to show that

$$\left\|(\mathcal{K}^{\lambda_n} - \mathcal{K}^\lambda)(\mathcal{A}^\lambda + \mathcal{K}^\lambda + i)^{-1}\right\|_{\mathfrak{B}(\mathfrak{H})} \to 0 \text{ as } n \to \infty. \tag{2.1}$$

We observe that $\mathcal{A}^\lambda + \mathcal{K}^\lambda$ is self-adjoint with the same domain as $\mathcal{A}^\lambda$ by the Kato-Rellich theorem, so $\mathcal{P}$ is also relatively compact with respect to $\mathcal{A}^\lambda + \mathcal{K}^\lambda$. By assumption iv) we have

$$(\mathcal{K}^{\lambda_n} - \mathcal{K}^\lambda)(\mathcal{A}^\lambda + \mathcal{K}^\lambda + i)^{-1} = (\mathcal{K}^{\lambda_n} - \mathcal{K}^\lambda)\mathcal{P}(\mathcal{A}^\lambda + \mathcal{K}^\lambda + i)^{-1}.$$

This is a composition of a strongly convergent sequence of operators and the compact operator $\mathcal{P}(\mathcal{A}^\lambda + \mathcal{K}^\lambda + i)^{-1}$. The compactness converts the strong convergence to norm convergence and proves (2.1). □

Remark 9. The operator $\mathcal{P}$ was key to the proof: it is required to obtain convergence $v_n \to v$ rather than simply $\mathcal{K}^{\lambda_n} v_n \to w$. In general, it is not true that if $\mathcal{T}_n \xrightarrow{s} \mathcal{T}_\infty$ and if for any $u \in \mathfrak{H}$ the closure of the set $\{\mathcal{T}_n u\}_{n\in\overline{\mathbb{N}}}$ is compact then $\mathcal{T}_n \to \mathcal{T}_\infty$. Consider for example $\mathcal{T}_n u = \langle e_n, u\rangle\, e_1$ where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathfrak{H}$. Then $\mathcal{T}_n \xrightarrow{s} 0 = \mathcal{T}_\infty$ and the set $\{\mathcal{T}_n u\}_{n\in\overline{\mathbb{N}}}$ is compact. However, consider for instance the sequence $\|\mathcal{T}_n e_n\| = 1$. Since $\{e_n\}_{n\in\mathbb{N}}$ form an orthonormal basis, this implies that norm convergence does not hold.

3 Constructing approximations

We first treat approximations of operators with discrete spectra, which are naturally defined via a sequence of increasing projection operators. For brevity, we call these approximations *n-approximations* ("n" refers to the dimension of the projection). Our strategy when treating operators with a continuous spectrum is to first "perturb" them by adding a family of unbounded operators (think of adding an unbounded potential to a Laplacian) depending upon a small parameter ε. For each $\varepsilon > 0$ these perturbations are assumed to eliminate any continuous spectrum, so that then we may apply an n-approximation. We therefore call these (ε, n)*-approximations*. We start with a standard result for which we could not find a good reference and we therefore state and prove it here.

Lemma 10. *Let $\mathfrak{H}$ be a Hilbert space and let $\mathcal{T}_n \xrightarrow{s.r.} \mathcal{T}$ as $n \to \infty$ with $\mathcal{T}_n, \mathcal{T}$ selfadjoint operators on $\mathfrak{H}$. Let $\mathcal{K}_n \xrightarrow{s} \mathcal{K}$ as $n \to \infty$ with $\mathcal{K}_n, \mathcal{K}$ bounded symmetric operators on $\mathfrak{H}$. Then $\mathcal{T}_n + \mathcal{K}_n$ and $\mathcal{T} + \mathcal{K}$ are self-adjoint and $\mathcal{T}_n + \mathcal{K}_n \xrightarrow{s.r.} \mathcal{T} + \mathcal{K}$.*

Proof. The self-adjointness follows from the Kato-Rellich theorem. For the convergence it is sufficient to prove that $(\mathcal{T}_n+\mathcal{K}_n+\alpha i)^{-1} \xrightarrow{s} (\mathcal{T}+\mathcal{K}+\alpha i)^{-1}$ for some real $\alpha \neq 0$. As the $\mathcal{K}_n$ are strongly convergent, by the uniform boundedness principle they are uniformly bounded in operator norm by some $M \geq \|\mathcal{K}\|_{\mathfrak{B}(\mathfrak{H})}$. Letting $\alpha = 2M$, and using the second Neumann series,

$$\begin{aligned}(\mathcal{T}_n+\mathcal{K}_n+\alpha i)^{-1} &= (\mathcal{T}_n+\alpha i)^{-1}(1+\mathcal{K}_n(\mathcal{T}_n+\alpha i)^{-1})^{-1} \\ &= (\mathcal{T}_n+\alpha i)^{-1}\sum_{k=0}^{\infty}(-1)^k(\mathcal{K}_n(\mathcal{T}_n+\alpha i)^{-1})^k\end{aligned}$$

is convergent uniformly in n as $\left\|\mathcal{K}_n(\mathcal{T}_n+\alpha i)^{-1}\right\|_{\mathfrak{B}(\mathfrak{H})} \leq M/\alpha = 1/2 < 1$. As $n \to \infty$ each term of the series converges strongly to the corresponding term of the series for $(\mathcal{T}+\mathcal{K}+\alpha i)^{-1}$ and as the series convergences uniformly in n we may may swap the order of summation and taking strong limits. $\square$

3.1 Operators with discrete spectra

In this paragraph we assume that $\mathcal{A}^\lambda$ has discrete spectrum and compact resolvent for some λ (and, in fact, for all λ, as $\mathcal{A}^\lambda$ is a holomorphic family of type (B)[2]). We exploit a property of self-adjoint holomorphic families [5, VII Theorem 3.9 and VII Remark 4.22]: the eigenvalues $\{\mu_k^\lambda\}_{k\in\mathbb{N}}$ and associated normalised eigenfunctions $\{e_k^\lambda\}_{k\in\mathbb{N}}$ of $\mathcal{A}^\lambda$ are holomorphic functions of $\lambda \in [0,1]$. An immediate consequence is that the unitary operator defined by

$$\begin{aligned}\mathcal{U}_\sigma^\lambda &: \mathfrak{H} \to \mathfrak{H} \\ e_k^\sigma &\mapsto e_k^\lambda \qquad \text{for any } k \in \mathbb{N}\end{aligned}$$

is jointly holomorphic in $\lambda, \sigma \in [0,1]$. We now define the n-truncation operator by

$$\begin{aligned}\mathcal{G}_n^\lambda &: \mathfrak{H} \to \mathfrak{H} \\ e_k^\lambda &\mapsto \begin{cases} e_k^\lambda & \text{if } k \leq n, \\ 0 & \text{if } k > n.\end{cases}\end{aligned}$$

Since the eigenfunctions form a complete orthonormal set we have the convergence $\mathcal{G}_n^\lambda \xrightarrow{s} 1$ as $n \to \infty$ for fixed λ. Additionally by expressing $\mathcal{G}_n^\lambda = \mathcal{U}_\sigma^\lambda \mathcal{G}_n^\sigma \mathcal{U}_\lambda^\sigma$ for some fixed $\sigma \in [0,1]$ we see that $\mathcal{G}_n^\lambda$ is jointly strongly continuous in n and λ.

[2] We remind the reader that the definition of a holomorphic family of type (B) is provided in subsection 1.2.

We now define the finite-dimensional approximations of $\mathcal{A}^\lambda$ and $\mathcal{M}^\lambda$ by

$$\mathcal{A}_n^\lambda = \mathcal{G}_n^\lambda \mathcal{A}^\lambda \mathcal{G}_n^\lambda \quad \text{and} \quad \mathcal{M}_n^\lambda = \mathcal{G}_n^\lambda \mathcal{M}^\lambda \mathcal{G}_n^\lambda, \tag{3.1}$$

respectively. It is too much to hope for convergence $\mathcal{M}_n^\lambda \xrightarrow{n.r.} \mathcal{M}^\lambda$ as $n \to \infty$, but we can hope for $\mathcal{M}_n^\lambda \xrightarrow{s.r.} \mathcal{M}^\lambda$. Indeed:

Lemma 11. *The family $\{\mathcal{M}_n^\lambda\}_{\lambda\in[0,1],n\in\overline{\mathbb{N}}}$ is continuous in the strong resolvent sense, where we use the convention that $\mathcal{M}_\infty^\lambda := \mathcal{M}^\lambda$.*

Proof. We need to show that if $\lambda_k \to \sigma \in [0,1]$ as $k \to \infty$, then

1. if $m_k \to m \in \mathbb{N}$, then $\mathcal{M}_{m_k}^{\lambda_k} \xrightarrow{s.r.} \mathcal{M}_m^\sigma$, and
2. if $m_k \to \infty$, then $\mathcal{M}_{m_k}^{\lambda_k} \xrightarrow{s.r.} \mathcal{M}^\sigma$.

For $m \in \mathbb{N}$ the result is obvious, so we may take $m = \infty$. By the stability of strong resolvent continuity with respect to bounded strongly continuous perturbations (see Lemma 10), it is sufficient to prove that $\mathcal{A}_{m_k}^{\lambda_k} \xrightarrow{s.r.} \mathcal{A}^\sigma$ as $k \to \infty$ and that $\mathcal{G}_{m_k}^{\lambda_k} \mathcal{K}^{\lambda_k} \mathcal{G}_{m_k}^{\lambda_k} \xrightarrow{s} \mathcal{K}^\sigma$. The latter is true as it is the composition of strong convergences. For the former it is sufficient to show that $(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1} \xrightarrow{s} (\mathcal{A}^\sigma + i)^{-1}$ as $k \to \infty$. Splitting this term as

$$(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1} = \mathcal{G}_{n_k}^{\lambda_k}(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1} + (1 - \mathcal{G}_{n_k}^{\lambda_k})(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1},$$

we see that since $(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1}$ is uniformly bounded the second part converges strongly to zero by the convergence $\mathcal{G}_{n_k}^{\lambda_k} \xrightarrow{s} 1$. For the first part, since $\mathcal{G}_{n_k}^{\lambda_k}$ is a spectral projection associated with $\mathcal{A}^{\lambda_k}$ we have

$$\mathcal{G}_{n_k}^{\lambda_k}(\mathcal{A}_{n_k}^{\lambda_k} + i)^{-1} = \mathcal{G}_{n_k}^{\lambda_k}(\mathcal{A}^{\lambda_k} + i)^{-1}\mathcal{G}_{n_k}^{\lambda_k}$$

which converges to $(\mathcal{A}^\sigma + i)^{-1}$ by the composition of strong convergences. □

3.2 Operators with continuous spectra

We are now ready to turn to the general case of families $\{\mathcal{A}^\lambda\}_{\lambda\in[0,1]}$ that may have continuous spectra. Such operators require (ε, n)-approximations. The ε-approximations $\mathcal{A}_\varepsilon^\lambda$ of $\mathcal{A}^\lambda$ were defined in (1.3) and the corresponding approximations $\mathcal{M}_\varepsilon^\lambda$ were defined in (1.5).

Lemma 12. *1. For any $\varepsilon > 0$, $\{\mathcal{A}_\varepsilon^\lambda\}_{\lambda\in D_0}$ is a holomorphic family of type (B) with compact resolvent.*

2. For any $\lambda \in [0,1], \varepsilon \geq 0$, $\mathcal{A}_\varepsilon^\lambda$ is self-adjoint and we have $\mathcal{A}_\varepsilon^\lambda \geq \mathcal{A}^\lambda \geq 1 + \alpha$.

Proof. The second claim is obvious since $\mathcal{W}^\lambda \geq 0$. For the first we must show $\mathfrak{a}_\varepsilon^\lambda$ is sectorial and that its domain $\mathfrak{D}(\mathfrak{a}_\varepsilon^\lambda)$ is independent of λ and dense in $\mathfrak{H}$, and that for any fixed $u \in \mathfrak{D}(\mathfrak{a}_\varepsilon^\lambda)$ the function $\mathfrak{a}_\varepsilon^\lambda[u]$ is holomorphic in $\lambda \in D_0$. For any $\lambda \in D_0$, $\mathfrak{a}_\varepsilon^\lambda$ is the sum of sectorial forms $\mathfrak{a}^\lambda$ and $\varepsilon\mathfrak{w}^\lambda$ so by [5, VI§1.6-Theorem 1.33] it is closed and sectorial with domain $\mathfrak{D}(\mathfrak{a}) \cap \mathfrak{D}(\mathfrak{w}^\lambda)$, which is independent of λ since $\mathcal{A}^\lambda, \mathcal{W}^\lambda$ are holomorphic families of type (B). Furthermore, we assumed that $\mathfrak{D}(\mathfrak{a}) \cap \mathfrak{D}(\mathfrak{w}^\lambda)$ is dense in $\mathfrak{H}$. For any fixed $u \in \mathfrak{D}(\mathfrak{a}_\varepsilon^\lambda)$, $\mathfrak{a}_\varepsilon^\lambda[u] = \mathfrak{a}^\lambda[u] + \varepsilon\mathfrak{w}^\lambda[u]$ is the sum of two holomorphic functions of $\lambda \in D_0$, so $\mathfrak{a}_\varepsilon^\lambda[u]$ is also holomorphic in D_0. Finally by the assumption that the inclusion from $(\mathfrak{D}(\mathfrak{a}_\varepsilon^\lambda), \|\cdot\|_{\mathfrak{a}_\varepsilon^\lambda})$ into $\mathfrak{H}$ is compact we deduce that the resolvent of $\mathcal{A}_\varepsilon^\lambda$ is compact. □

For each $\varepsilon > 0$ the operator $\mathcal{A}_\varepsilon^\lambda$ has a discrete spectrum, and therefore the n-approximations of $\mathcal{A}_\varepsilon^\lambda$ and $\mathcal{M}_\varepsilon^\lambda$ may be defined analogously to (3.1) via the projection operators

$$\begin{aligned} \mathcal{G}_{\varepsilon,n}^\lambda &: \mathfrak{H} \to \mathfrak{H} \\ e_{\varepsilon,k}^\lambda &\mapsto \begin{cases} e_{\varepsilon,k}^\lambda & \text{if } k \leq n, \\ 0 & \text{if } k > n, \end{cases} \end{aligned}$$

(where $\{e_{\varepsilon,k}^\lambda\}_{k\in\mathbb{N}}$ are normalised eigenfunctions of $\mathcal{A}_\varepsilon^\lambda$) as

$$\mathcal{A}_{\varepsilon,n}^\lambda = \mathcal{G}_{\varepsilon,n}^\lambda \mathcal{A}_\varepsilon^\lambda \mathcal{G}_{\varepsilon,n}^\lambda \quad \text{and} \quad \mathcal{M}_{\varepsilon,n}^\lambda = \mathcal{G}_{\varepsilon,n}^\lambda \mathcal{M}_\varepsilon^\lambda \mathcal{G}_{\varepsilon,n}^\lambda.$$

We know by Lemma 11 that the family $\{\mathcal{A}_{\varepsilon,n}^\lambda\}_{\lambda\in[0,1],n\in\overline{\mathbb{N}}}$ is continuous in the strong resolvent sense. In addition, we have:

Lemma 13. *The family $\{\mathcal{A}_\varepsilon^\lambda\}_{\lambda\in[0,1],\varepsilon\in[0,\infty)}$ is continuous in the strong resolvent sense.*

Proof. By the equivalence of strong and weak convergence of the resolvent for self-adjoint operators it is sufficient to prove that $(\mathcal{A}_\varepsilon^\lambda + 1)^{-1}$ is weakly continuous jointly in λ and ε (note that we evaluate the resolvent at -1 rather than at i due to the sectoriality of the operators away from the real line). Let $U \subseteq D_0$ be an open set containing the interval $[0,1]$ such that for $\lambda \in U$, $\operatorname{Re}\mathfrak{a}^\lambda \geq 1$ and $\operatorname{Re}\mathfrak{w}^\lambda \geq -1$. Such a set exists by the holomorphicity of the two families. Writing $\mathfrak{a}_\varepsilon^\lambda[f] + \sigma\|f\|^2 = \langle f, u\rangle$ where $f = (\mathcal{A}_\varepsilon^\lambda + \sigma)^{-1}u$ and taking the real part, we use the aforementioned bounds to obtain

$$\sup_{\varepsilon\in[0,1],\lambda\in U} \left\|(\mathcal{A}_\varepsilon^\lambda + 1)^{-1}\right\|_{\mathfrak{B}(\mathfrak{H})} \leq 1. \tag{3.2}$$

Now fix $u, v \in \mathfrak{H}$, let $\varepsilon_n \to \varepsilon_\infty \in [0, \infty)$ and define the sequence of holomorphic functions $f_n(\lambda) : U \to \mathbb{C}$ by

$$f_n(\lambda) = \left\langle (\mathcal{A}^\lambda_{\varepsilon_n} + 1)^{-1}u - (\mathcal{A}^\lambda_{\varepsilon_\infty} + 1)^{-1}u, v \right\rangle$$

with $f_\infty = 0$. To prove the joint weak continuity of the resolvent it is clearly sufficient to show that $f_n \to 0$ uniformly over $\lambda \in [0, 1]$. The case $\varepsilon_\infty > 0$ is straightforward so we assume that $\varepsilon_\infty = 0$. Without loss of generality we may assume that $\varepsilon_n \neq 0$ for all n. We will use a simple corollary of Montel's theorem that states that a sequence of holomorphic functions that is uniformly bounded on an open set $U \subseteq \mathbb{C}$ and converges pointwise in U converges uniformly on any compact set $K \subset U$. The uniform boundedness of f_n follows from (3.2) above. Thus it is suffices to show that $f_n \to 0$ pointwise. To this end we will establish pointwise convergence of the corresponding forms $\mathfrak{a}^\lambda_{\varepsilon_n}$. Indeed,

$$\forall \lambda \in D_0, w \in \mathfrak{D}(\mathfrak{a}^\lambda_{\varepsilon_n}), \quad \mathfrak{a}^\lambda_{\varepsilon_n}[w] - \mathfrak{a}^\lambda[w] = \varepsilon_n \mathfrak{w}^\lambda[w] \to 0 \quad \text{as } n \to \infty.$$

For $n \in \mathbb{N}$ the forms have common form domain $\mathfrak{D}(\mathfrak{a}) \cap \mathfrak{D}(\mathfrak{w})$, which is a form core for $\mathfrak{a}^\lambda$, and the sequence of form differences $\mathfrak{a}^\lambda_{\varepsilon_n} - \mathfrak{a}^\lambda$ is uniformly sectorial. Thus [5, VIII.§3.2-Theorem 3.6] applies, giving $\mathcal{A}^\lambda_{\varepsilon_n} \xrightarrow{s.r.} \mathcal{A}^\lambda$ as $n \to \infty$, which implies the pointwise convergence $f_n \to 0$ and completes the proof. $\square$

Corollary 14. *The family* $\{\mathcal{M}^\lambda_\varepsilon\}_{\lambda \in [0,1], \varepsilon \in [0,\infty)}$ *is continuous in the strong resolvent sense.*

Proof. This follows from the stability of strong resolvent continuity with respect to bounded strongly continuous perturbations. $\square$

4 Proof of Theorem 2′

4.1 Compactness results

Let $\mathcal{M}^\lambda$ be a family of operators as defined (1.4) and let $\mathcal{M}^\lambda_\varepsilon$ and $\mathcal{M}^\lambda_{\varepsilon,n}$ be the corresponding ε and (ε, n)-approximations as defined in section 3. We now show that these approximations are well-behaved, in the following sense:

Proposition 15. *Define the set* $\Delta = \mathfrak{D}(\mathcal{M}^\lambda_\varepsilon) \times (-\infty, 1] \times [0, 1]$. *Fix* $\varepsilon^* > 0$. *Then the set of eigenfunctions*

$$\mathfrak{A} = \{(u, \sigma, \lambda, \varepsilon) \in \Delta \times [0, \varepsilon^*] : \|u\| = 1, \mathcal{M}^\lambda_\varepsilon u = \sigma u\}$$

is compact. In addition, for any fixed $\varepsilon > 0$ the set of approximated eigenfunctions

$$\mathfrak{A}_\varepsilon = \{(u, \sigma, \lambda, n) \in \Delta \times \overline{\mathbb{N}} : \|u\| = 1, u = \mathcal{G}^\lambda_{\varepsilon,n} u, \mathcal{M}^\lambda_{\varepsilon,n} u = \sigma u\}$$

is relatively compact.

We will first prove a slightly more general result:

Lemma 16. *Fix $\varepsilon^* > 0$ and define the set*

$$\mathfrak{A}' = \{(u, \sigma, \lambda, \varepsilon, n) \in \Delta \times [0, \varepsilon^*] \times \overline{\mathbb{N}} : \|u\| = 1, u = \mathcal{G}^\lambda_{\varepsilon,n} u, \mathcal{M}^\lambda_{\varepsilon,n} u = \sigma u\}.$$

Let $\{(u_k, \sigma_k, \lambda_k, \varepsilon_k, n_k)\}_{k=1}^\infty$ be a sequence in $\mathfrak{A}'$ with $\lambda_k \to \lambda, \sigma_k \to \sigma, \varepsilon_k \to \varepsilon, n_k \to n$ as $k \to \infty$. Then the sequence has a convergent subsequence if $\mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k}$ has a strong limit as $k \to \infty$.

Proof. Each u_k solves the equation

$$\mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} \mathcal{A}^{\lambda_k}_{\varepsilon_k} \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} u_k - \sigma_k u_k + \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} \mathcal{K}^{\lambda_k} \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} u_k = 0.$$

The requirement that $u_k = \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} u_k$ and the fact that $\mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k}$ commutes with $\mathcal{A}^{\lambda_k}_{\varepsilon_k}$ means that this is equivalent to

$$\mathcal{A}^{\lambda_k}_{\varepsilon_k} u_k - \sigma_k u_k + \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} \mathcal{K}^{\lambda_k} u_k = 0. \tag{4.1}$$

Taking the inner product with u_k we estimate,

$$\mathfrak{a}^0[u_k] \le C \mathfrak{a}^{\lambda_k}[u_k] \le C \mathfrak{a}^{\lambda_k}_{\varepsilon_k}[u_k] \le C\sigma_k \|u_k\|^2 + C \sup_{\lambda \in [0,1]} \left\| \mathcal{K}^\lambda \right\|_{\mathfrak{B}(\mathfrak{H})} \|u_k\|^2 \le C' \tag{4.2}$$

where C is independent of k comes from the relative form boundedness of the holomorphic family $\{\mathcal{A}^\lambda\}_{\lambda \in D_0}$ (see [5, VII-§4.2]) and the supremum is finite by the uniform boundedness principle as $\{\mathcal{K}^\lambda\}_{\lambda \in [0,1]}$ is strongly continuous. Hence by the relative form compactness of $\mathcal{P}$ to $\mathfrak{a}^0$ we may pass to a subsequence (though we retain the subscript k) for which

$$\mathcal{P} u_k \to v \in \mathfrak{H}.$$

Then by rewriting (4.1) and using $\mathcal{K}^\lambda = \mathcal{K}^\lambda \mathcal{P}$ for all $\lambda \in [0, 1]$ we have

$$u_k = -(\mathcal{A}^{\lambda_k}_{\varepsilon_k} - \sigma_k)^{-1} \mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k} \mathcal{K}^{\lambda_k} \mathcal{P} u_k \tag{4.3}$$

where the resolvent exists by the assumption that $\mathcal{A}^\lambda \ge 1 + \alpha$ for all $\lambda \in [0, 1]$. Under the assumption that $\mathcal{G}^{\lambda_k}_{\varepsilon_k,n_k}$ converges strongly to some bounded operator $\mathcal{G}$ as $k \to \infty$ we then have

$$u_k \to -(\mathcal{A}^\lambda_\varepsilon - \sigma)^{-1} \mathcal{G} \mathcal{K}^\lambda v$$

so that u_k is a convergent subsequence. $\square$

Lemma 17. *The spectrum of the operator $\mathcal{M}_\varepsilon^\lambda$ is bounded below uniformly in $\lambda \in [0,1]$ and $\varepsilon \in [0,\infty)$.*

Proof. It suffices to bound the numerical range. Let $u \in \mathfrak{D}(\mathcal{M}_\varepsilon^\lambda)$ with $\|u\| = 1$ then

$$\mathfrak{m}_\varepsilon^\lambda[u] = \mathfrak{a}_\varepsilon^\lambda[u] + \left\langle \mathcal{K}^\lambda u, u \right\rangle \geq \mathfrak{a}^\lambda[u] - \sup_{\lambda\in[0,1]} \left\| \mathcal{K}^\lambda \right\|_{\mathfrak{B}(\mathfrak{H})} \geq 1 + \alpha - \sup_{\lambda\in[0,1]} \left\| \mathcal{K}^\lambda \right\|_{\mathfrak{B}(\mathfrak{H})}$$

where the supremum is finite by the uniform boundedness principle. □

Now we are ready to prove Proposition 15:

Proof of Proposition 15. We first note that we can interpret $\mathfrak{A}$ and $\mathfrak{A}_\varepsilon$ as subsets of $\mathfrak{A}'$ by

$$\mathfrak{A} \ni (u, \sigma, \lambda, \varepsilon) \mapsto (u, \sigma, \lambda, \varepsilon, \infty) \in \mathfrak{A}'$$
$$\mathfrak{A}_\varepsilon \ni (u, \sigma, \lambda, n) \mapsto (u, \sigma, \lambda, \varepsilon, n) \in \mathfrak{A}'$$

Let $\{(u_k, \sigma_k, \lambda_k, \varepsilon_k, n_k)\}_{k=1}^\infty$ be a sequence in $\mathfrak{A}'$. By Lemma 17 the σ_k are relatively compact and similarly $\lambda_k \in [0,1], n_k \in \overline{\mathbb{N}}$ and $\varepsilon_k \in [0, \varepsilon^*]$ are relatively compact and we may pass to a subsequence (maintaining the index k) for which $\lambda_k \to \lambda, \sigma_k \to \sigma, \varepsilon_k \to \varepsilon, n_k \to n$ as $k \to \infty$. Hence Lemma 16 is applicable, and to show a convergent subsequence we must show that $\mathcal{G}_{\varepsilon_k, n_k}^{\lambda_k}$ has a strong limit as $k \to \infty$. On the one hand if the original sequence was inside $\mathfrak{A} \subset \mathfrak{A}'$ then we have $n_k = \infty$ for all k. Hence $\mathcal{G}_{\varepsilon_k, n_k}^{\lambda_k} = 1$ by definition. On the other hand if the original sequence was inside $\mathfrak{A}_\varepsilon \subset \mathfrak{A}'$ then $\varepsilon_k = \varepsilon > 0$ for all k, so that as remarked before $\mathcal{G}_{\varepsilon,n}^\lambda$ is jointly strongly continuous in λ, n so that $\mathcal{G}_{\varepsilon,n_k}^{\lambda_k} \xrightarrow{s} \mathcal{G}_{\varepsilon,n}^\lambda$ as $k \to \infty$. □

4.2 Convergence of spectra

We can now use the above compactness results together with the continuity results to prove Theorem 2′.

Proof of Theorem 2′. We will prove that each of Σ and Σ_ε are both upper semi-continuous and lower semi-continuous. The lower semi-continuity of spectra under strong resolvent convergence of self-adjoint operators is standard (e.g. [5, VIII.§1.2-Theorem 1.14.]). As we have that $\mathcal{M}_\varepsilon^\lambda$ is continuous in the strong resolvent sense (Lemma 13) we have that Σ is lower semi-continuous. For Σ_ε we must be slightly more careful due to the spurious eigenvalue of $\mathcal{M}_{\varepsilon,n}^\lambda$ at 0 for $n \neq \infty$ (see Remark 5 for further discussion of this eigenvalue, as well as the definition of $\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda$ which shall appear below).

We instead consider the operator $\widehat{\mathcal{M}}^\lambda_{\varepsilon,n} := \mathcal{M}^\lambda_{\varepsilon,n} + M(1 - \mathcal{G}^\lambda_{\varepsilon,n}) : \mathfrak{H} \to \mathfrak{H}$ where $M > 1$ is arbitrary (note that $\widehat{\mathcal{M}}^\lambda_{\varepsilon,\infty} = \mathcal{M}^\lambda_{\varepsilon,\infty}$). This moves the spurious eigenvalue to $M \notin (-\infty, 1]$. By Lemma 11 the family $\{\mathcal{M}^\lambda_{\varepsilon,n}\}_{\lambda\in[0,1],n\in\overline{\mathbb{N}}}$ is continuous in the strong resolvent sense, and using the stability of strong resolvent convergence with respect to strongly continuous bounded perturbations $\{\widehat{\mathcal{M}}^\lambda_{\varepsilon,n}\}_{\lambda\in[0,1],n\in\overline{\mathbb{N}}}$ is also continuous in the strong resolvent sense. Moreover, the spectra of $\widehat{\mathcal{M}}^\lambda_{\varepsilon,n}$ and $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ agree in $(-\infty, 1]$ as $M > 1$, which establishes the lower semi-continuity of Σ_ε.

For the upper semi-continuity we shall use the compactness result Proposition 15. As the proof for Σ is slightly simpler than that for Σ_ε and otherwise the same we shall leave it to the reader. Let $\lambda_k \to \lambda \in [0,1]$, $n_k \to n \in \overline{\mathbb{N}}$ and $\sigma_k \to \sigma \in (-\infty, 1)$ as $k \to \infty$ with σ_k an eigenvalue of $\widetilde{\mathcal{M}}^{\lambda_k}_{\varepsilon,n_k}$. Then it is sufficient to prove that σ is an eigenvalue of $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$. Let u_k be the normalised eigenfunctions. Then $\{(u_k, \sigma_k, n_k, \lambda_k)\}_{k=1}^\infty \subset \mathfrak{A}_\varepsilon$ is a compact set. Hence we may pass to a subsequence (still indexed with k) for which $u_k \to u$. Then by $\widetilde{\mathcal{M}}^{\lambda_k}_{\varepsilon,n_k} - \sigma_k \xrightarrow{s.r.} \widetilde{\mathcal{M}}^\lambda_{\varepsilon,n} - \sigma$ as $k \to \infty$ we see that u is an eigenfunction associated with the eigenvalue σ. Indeed, if we have some self-adjoint operators $\mathcal{T}_k \xrightarrow{s.r.} \mathcal{T}$ and elements $z_k \to z$ with $\mathcal{T}_k z_k = 0$ then

$$\begin{aligned}
\mathcal{T}_k z_k &= 0 && \iff \\
(\mathcal{T}_k + i) z_k &= i z_k && \iff \\
z_k &= i(\mathcal{T}_k + i)^{-1} z_k \\
&\downarrow \text{ as } k \to \infty \text{ by } \mathcal{T}_k \xrightarrow{s.r.} \mathcal{T} \\
z &= i(\mathcal{T} + i)^{-1} z && \iff \\
\mathcal{T} z &= 0.
\end{aligned}$$

□

5 Non-positive operators: proof of Theorem 2

We define the ε-approximations of $\mathcal{A}^\lambda_\pm$ as before in terms of a pair of holomorphic families $\mathcal{W}^\lambda_\pm$ with the same assumptions. The eigenprojections of $\mathcal{A}^\lambda_\varepsilon$ are then denoted by $\mathcal{G}^\lambda_{\pm,\varepsilon,n}$ and we define

$$\mathcal{G}^\lambda_{\varepsilon,n} = \begin{bmatrix} \mathcal{G}^\lambda_{+,\varepsilon,n} & 0 \\ 0 & \mathcal{G}^\lambda_{-,\varepsilon,n} \end{bmatrix}$$

and

$$\begin{aligned}
\mathcal{A}^\lambda_{\varepsilon,n} &= \mathcal{G}^\lambda_{\varepsilon,n} \mathcal{A}^\lambda_\varepsilon \mathcal{G}^\lambda_{\varepsilon,n} \\
\mathcal{M}^\lambda_{\varepsilon,n} &= \mathcal{G}^\lambda_{\varepsilon,n} \mathcal{M}^\lambda_\varepsilon \mathcal{G}^\lambda_{\varepsilon,n}.
\end{aligned}$$

All the preceding proofs of continuity can be adapted to this case. Indeed, Proposition 8 holds without modification, while Lemma 11 and Lemma 13 can be extended by using the identity

$$\left(\begin{bmatrix} \mathcal{T}_+ & 0 \\ 0 & \mathcal{T}_- \end{bmatrix} + i\right)^{-1} = \begin{bmatrix} (\mathcal{T}_+ + i)^{-1} & 0 \\ 0 & (\mathcal{T}_- + i)^{-1} \end{bmatrix}$$

and the stability of norm (resp. strong) continuity to symmetric bounded norm (reps. strongly) continuous perturbations. The compactness and spectral continuity results need more modification. Recall that the discrete region of the spectrum is the gap $(-\alpha - 1, 1 + \alpha)$ rather than the half-line $(-\infty, 1 + \alpha)$. We restate this below:

Proposition 18. *Fix $\varepsilon^* > 0$ and let $\Delta = \mathfrak{D}(\mathcal{M}_\varepsilon^\lambda) \times [-1, 1] \times [0, 1]$. Then the set of eigenfunctions*

$$\mathfrak{A} = \{(u, \sigma, \lambda, \varepsilon) \in \Delta \times [0, \varepsilon^*] : \|u\| = 1, \mathcal{M}_\varepsilon^\lambda u = \sigma u\}$$

is compact. Let $\varepsilon > 0$ be fixed then set of approximated eigenfunctions

$$\mathfrak{A}_\varepsilon = \{(u, \sigma, \lambda, n) \in \Delta \times \overline{\mathbb{N}} : \|u\| = 1, u = \mathcal{G}_{\varepsilon,n}^\lambda u, \mathcal{M}_{\varepsilon,n}^\lambda u = \sigma u\}$$

is relatively compact.

Proof (sketched). To prove the compactness results we use a version of Lemma 16 with $\sigma \in (-\infty, 1]$ replaced with $\sigma \in [-1, 1]$ in the definition of $\mathfrak{A}'$. Once we have this the proof is identical to Proposition 15. In the proof of Lemma 16 we need only change (4.2) to the two estimates

$$\begin{aligned} \mathfrak{a}_\pm^0[u_k^\pm] &\le C_\pm \mathfrak{a}_\pm^{\lambda_k}[u_k^\pm] \le C_\pm \mathfrak{a}_{\pm,\varepsilon_k}^{\lambda_k}[u_k^\pm] \\ &\le C_\pm |\sigma_k| \left\|u_k^\pm\right\|^2 + C_\pm \sup_{\lambda \in [0,1]} \left\|\mathcal{K}^\lambda\right\|_{\mathfrak{B}(\mathfrak{H})} \|u_k\|^2 \le C' \end{aligned}$$

obtained by taking the inner product of (4.1) with $u_k^\pm$ where $u_k = (u_k^+, u_k^-) \in \mathfrak{H}_+ \times \mathfrak{H}_-$, from which the relative compactness of $\mathcal{P}u_k$ follows, and lastly note that $\mathcal{A}_\pm^\lambda \ge 1 + \alpha$ implies that the resolvent $(\mathcal{A}_{\varepsilon_k}^{\lambda_k} - \sigma_k)^{-1}$ exists in (4.3). $\square$

With this result the continuity of spectra carries over identically as before, except for the different region of the spectrum considered (previously $(-\infty, 1)$, now $(-1, 1)$). This proves Theorem 2.

6 An application: plasma instabilities

The discussion in this section is informal. As stability analysis typically relies on a detailed understanding of the spectrum of the linearised problem, most results in this direction require delicate spectral analysis. However, an outstanding open problem has been stability analysis of plasmas that do not possess special symmetries (such as periodicity or monotonicity[3]) due to the more complicated structure of the spectrum. A significant obstacle has been the existence of an essential spectrum extending to both $\pm\infty$. Let us briefly outline the problem, which is treated in detail in [2].

Plasmas are typically modelled by the relativistic Vlasov-Maxwell system: Letting $f = f(t, x, v)$ be a function measuring the density of electrons that at time $t \geq 0$ are located at the point $x \in \mathbb{R}^d$, have momentum $v \in \mathbb{R}^d$ and velocity $\hat{v} = v/\sqrt{1 + |v|^2}$, the Vlasov equation

$$\frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f + \mathbf{F} \cdot \nabla_v f = 0 \tag{6.1}$$

is a transport equation describing their evolution due to the some force $\mathbf{F}$. Here we have taken the mass of the electrons and the speed of light to be 1 for simplicity. The forcing term $\mathbf{F}$ captures the physics of the problem, and in this case it is the Lorentz force

$$\mathbf{F} = -\mathbf{E} - \hat{v} \times \mathbf{B}$$

where $\mathbf{E} = \mathbf{E}(t, x)$ and $\mathbf{B} = \mathbf{B}(t, x)$ are the (self-consistent) electric and magnetic fields, respectively. They satisfy Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}, \tag{6.2}$$

where $\rho = \rho(t, x) = -\int f \; dv$ is the charge density and $\mathbf{j} = \mathbf{j}(t, x) = -\int \hat{v} f \; dv$ is the current density (negative signs are due to the electrons being negatively charged). Linearising (6.1) we obtain

$$\frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f + \mathbf{F^0} \cdot \nabla_v f = -\mathbf{F} \cdot \nabla_v f^0, \tag{6.3}$$

where f^0 and $\mathbf{F^0}$ are the equilibrium density and force field, respectively, and f and $\mathbf{F}$ are their first order perturbations. Maxwell's equations do not require linearisation as they are already linear. We seek solutions to (6.2)-(6.3) that grow exponentially in time. Therefore, substituting into (6.3) the ansatz that all time-dependent quantities behave like $e^{\lambda t}$ with $\lambda > 0$, we get

$$\lambda f + \hat{v} \cdot \nabla_x f + \mathbf{F^0} \cdot \nabla_v f = -\mathbf{F} \cdot \nabla_v f^0.$$

[3] *Monotonicity*, roughly speaking, means that there are fewer particles at higher energies. For a precise definition see e.g. [1].

An inversion of this equation leaves us with the integral expression

$$f = -(\lambda + (\hat{v}, \mathbf{F^0}) \cdot \nabla_{x,v})^{-1} (\mathbf{F} \cdot \nabla_v f^0) \tag{6.4}$$

which depends upon λ as a parameter. By substituting the expression (6.4) into Maxwell's equations (6.2), f is eliminated as an unknown, and the only unknowns left are the electromagnetic fields, expressed via the corresponding potentials, ϕ and $\mathbf{A}$. [Note that an immediate benefit is that the problem now only involves the spatial variable x, and not the full phase-space variables x, v]

We are therefore left with the task of showing that Maxwell's equations are satisfied with the parameter $\lambda > 0$. Gauss' equation, for instance, becomes

$$\Delta\phi = \nabla \cdot \mathbf{E} = \rho = -\int f \ dv = \int (\lambda + (\hat{v}, \mathbf{F^0}) \cdot \nabla_{x,v})^{-1} (\mathbf{F} \cdot \nabla_v f^0) \ dv$$

which is an equation of the form

$$\Delta\phi + \mathcal{K}^{\lambda}_{++}\phi + \mathcal{K}^{\lambda}_{+-}\mathbf{A} = 0. \tag{6.5}$$

The rest of Maxwell's equations can be written as

$$-\mathbf{\Delta}\mathbf{A} + \mathcal{K}^{\lambda}_{-+}\phi + \mathcal{K}^{\lambda}_{--}\mathbf{A} = \mathbf{0}. \tag{6.6}$$

This system for ϕ and $\mathbf{A}$ is precisely of the form (1.1). Exhibiting linear instability, i.e. the existence of a growing mode with rate $\lambda > 0$, is equivalent to solving this system for some $\lambda > 0$. This is done by tracking the spectrum of this (self-adjoint) problem as λ varies from 0 to $+\infty$, and showing that for some intermediate value λ_0 there is a non-trivial kernel. As already mentioned above, the fact that the spectrum extends to both $\pm\infty$ (due to the Laplacians with opposite signs) and is continuous (except for a gap around zero) makes this task difficult. Theorem 2 allows us to settle this problem, and we again refer to [2] for full details.

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