ERRATUM TO: “APPROXIMATIONS OF STRONGLY CONTINUOUS FAMILIES OF UNBOUNDED SELF-ADJOINT OPERATORS”

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Abstract. A gap in the proof of the original article is fixed. As a result, the formulation of the main theorem is modified accordingly.

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1. Introduction

The original article [BAH16] dealt with finite-dimensional symmetric approximations of families of self-adjoint operators of the form

$$\mathcal{M}^\lambda = \mathcal{A}^\lambda + \mathcal{K}^\lambda = \begin{bmatrix} -\Delta + \alpha(\lambda) & 0 \\ 0 & \Delta - \alpha(\lambda) \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\lambda_{++} & \mathcal{K}^\lambda_{+-} \\ \mathcal{K}^\lambda_{-+} & \mathcal{K}^\lambda_{--} \end{bmatrix}, \quad \lambda \in [0,1] \tag{1.1}$$

acting in an appropriate subspace of $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, where $\{\mathcal{K}^\lambda\}_{\lambda\in[0,1]}$ is a bounded, symmetric and strongly continuous family and $\alpha(\lambda) > \alpha > 1$ is continuous. The spectrum of $\mathcal{M}^\lambda$ was discretised by adding a potential, leading us to define

$$\mathcal{M}^\lambda_\varepsilon = \mathcal{A}^\lambda + \mathcal{K}^\lambda + \varepsilon\mathcal{W}^\lambda \tag{1.2}$$

which is assumed to have a compact resolvent for all $\varepsilon > 0$ (the precise details are omitted in this note). Finally, an $n \times n$ matrix $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ was defined by restricting $\mathcal{M}^\lambda_\varepsilon$ to a subspace spanned by n eigenfunctions of $\mathcal{A}^\lambda + \varepsilon\mathcal{W}^\lambda$ (chosen in an appropriate way). The main result – Theorem 3 – asserted that $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ recovers the spectrum of $\mathcal{M}^\lambda$ in $(-1,1)$ and moreover that the spectrum of $\widetilde{\mathcal{M}}^\lambda_{\varepsilon,n}$ converge uniformly in λ to the spectrum of $\mathcal{M}^\lambda$ on compact subsets of $(-1,1)$ as $\varepsilon \to 0$ and $n \to \infty$.

The purpose of this erratum is to correct this statement by taking into consideration the possible appearance of eigenvalues entering $(-1,1)$ at the boundary as λ varies (in other words, we lack upper-semicontinuity).

The possible discrepancy in the original statement stems from a gap in the proof: while the theorem treats the convergence of spectra in the *open* interval $(-1,1)$, the crucial compactness result meant to show upper-semicontinuity (Proposition 18) deals with the *closed* interval $[-1,1]$. We settle this discrepancy by considering a different topology. The approach of the original article was to think of the spectrum as a subset of the real line and measure distance according to the Hausdorff distance

$$d_H(X,Y) := \max\left(\sup_{y\in Y}\inf_{x\in X}|x-y|, \sup_{x\in X}\inf_{y\in Y}|x-y|\right), \qquad X,Y \subset \mathbb{R}.$$

Instead, we think of the spectrum as a *measure* (counting multiplicities) and we assess convergence in terms of *weak convergence of measures*. While eigenvalues are detected in both topologies, this new topology allows us to control eigenvalues entering at the boundary by introducing a smooth cutoff function (see (2.1) below). We recall that a sequence of finite Borel measures (on some Polish space $\mathcal{X}$) $\{\mu_n\}_{n\in\mathbb{N}}$ is said to converge to a measure μ weakly ($\mu_n \rightharpoonup \mu$) if $\int_{\mathcal{X}} f \, \mathrm{d}\mu_n \to \int_{\mathcal{X}} f \, \mathrm{d}\mu$ for any f that is bounded and continuous.

It is a general fact that for a separable topological vector space X, the closed unit ball of X^* is a separable metric space in the weak-* topology [Rud73, Theorem 3.16]. Therefore, there exists a metric on the space of measures with total mass ≤ 1, which is compatible with the weak topology of measures defined above (the conflict between functional analysts and probabilists in what is called the *weak* topology and what is called the *weak-** topology is well-known). We denote this metric d_{meas}, and note that it applies to any compact (in the weak-* topology) set of measures. In conclusion, we have that for any sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of uniformly bounded measures, and any finite measure μ,

$$d_{\text{meas}}(\mu_n, \mu) \to 0 \quad \iff \quad \mu_n \rightharpoonup \mu.$$

2. Reformulating the main theorem

In the original article we studied continuity properties (in the sense of the Hausdorff distance) of the two set-valued maps

$$\begin{aligned} \Sigma : [0,1] \times [0,\varepsilon^*] &\to (\text{closed subsets of } (-1,1), d_H) \\ \Sigma(\lambda,\varepsilon) &= (-1,1) \cap \operatorname{sp}(\mathcal{M}_\varepsilon^\lambda) \end{aligned}$$

and

$$\begin{aligned} \Sigma_\varepsilon : [0,1] \times \mathbb{N} &\to (\text{closed subsets of } (-1,1), d_H) \\ \Sigma_\varepsilon(\lambda,n) &= (-1,1) \cap \operatorname{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda). \end{aligned}$$

The main theorem in the paper was:

Original (flawed) theorem. The mappings $\Sigma(\cdot,\cdot)$ and $\Sigma_\varepsilon(\cdot,n)$ are continuous in their arguments, and as $n\to\infty$, $\Sigma_\varepsilon(\lambda,n) \to \Sigma(\lambda,\varepsilon)$ uniformly in $\lambda \in [0,1]$.

Instead, for $\lambda \in [0,1]$, $\varepsilon \geq 0$ we define the spectral measures

$$\nu_{\lambda,\varepsilon} = \sum_{x \in \operatorname{sp}_{\text{pp}}(\mathcal{M}_\varepsilon^\lambda) \setminus \operatorname{sp}_{\text{ess}}(\mathcal{M}_\varepsilon^\lambda)} \delta_x, \quad \textit{(counting multiplicities)}$$

(note that the essential spectrum is trivial whenever $\varepsilon > 0$) and for any $\varepsilon > 0$, $\lambda \in [0,1]$ and $n \in \mathbb{N}$ the spectral measures

$$\widetilde{\nu}_{\lambda,\varepsilon,n} = \sum_{x \in \operatorname{sp}(\widetilde{\mathcal{M}}_{\varepsilon,n}^\lambda)} \delta_x, \quad \textit{(counting multiplicities)},$$

where δ_x is the standard Dirac delta function centred at x. Consider a cutoff function φ satisfying

$$\varphi(x) = \begin{cases} 1 & x \in [-1,1] \\ 0 & x \in \mathbb{R} \setminus (-1 - \frac{\alpha-1}{2}, 1 + \frac{\alpha-1}{2}) \end{cases}, \quad \varphi \in C(\mathbb{R},[0,1]).$$

Finally, define the measures

$$\mu_{\lambda,\varepsilon}^\varphi = \varphi \nu_{\lambda,\varepsilon} \tag{2.1}$$

and

$$\widetilde{\mu}^{\varphi}_{\lambda,\varepsilon,n} = \varphi \widetilde{\nu}_{\lambda,\varepsilon,n}. \tag{2.2}$$

The main theorem may now be restated as:

Theorem 2.1. *The mappings* $[0,1]\times[0,\infty) \ni (\lambda,\varepsilon) \mapsto \mu^{\varphi}_{\lambda,\varepsilon}$ *and* $[0,1] \ni \lambda \mapsto \widetilde{\mu}^{\varphi}_{\lambda,\varepsilon,n}$ *(*$\varepsilon > 0$*,* $n \in \mathbb{N}$*) are weakly continuous. As* $n \to \infty$*,* $d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda,\varepsilon,n}, \mu^{\varphi}_{\lambda,\varepsilon}) \to 0$ *uniformly in* $\lambda \in [0,1]$.

Remark 2.2. i. The cutoff function φ is essential in that it allows eigenvalues to enter the interval $[-1,1]$ "gradually" as the parameters λ and ε vary. While it is indeed unfortunate that the statement above involves this auxiliary function, we point out that φ is identically 1 inside $[-1,1]$ (which is the interval of interest for us) and therefore does not play a role for the approximation problem there.

ii. Furthermore, remembering that our goal is to approximate eigenvalues, the new formulation contains more information than in the original paper as it captures multiplicities, a feature that the Hausdorff distance lacks. **We therefore emphasize that this new formulation is not weaker than the original one.** In fact, for measures on the real line, weak convergence is equivalent to pointwise convergence of the cumulative distribution function at its points of continuity [Vil03, Proposition 7.15].

Remark 2.3. From the results of the original paper we know that the following facts hold:

i. *Upper-semicontinuity:* If $[0,1]\times[0,+\infty) \ni (\lambda_m,\varepsilon_m) \to (\lambda_\infty,\varepsilon_\infty)$, $[-1-\alpha,1+\alpha] \ni \sigma_m \to \sigma_\infty$ and $\mathcal{M}^{\lambda_m}_{\varepsilon_m} u_m = \sigma_m u_m$ where $\|u_m\| = 1$ then u_m has a subsequence converging strongly to some $u_\infty \neq 0$ and $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty} u_\infty = \sigma_\infty u_\infty$. That is, we have upper-semicontinuity of the spectrum on the closed interval $[-1-\alpha,1+\alpha]$: subsequences of eigenvalues of $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ converge to all eigenvalues of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$.

ii. *Lower-semicontinuity:* The spectrum is lower-semicontinuous under strong resolvent perturbations. This implies that each eigenvalue of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ is a limit of eigenvalues of a subsequence of $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$.

Proof of Theorem 2.1. We split the proof into three parts, denoted **I, II, III**.

I. Claim: along any sequence $[0,1]\times[0,+\infty) \ni (\lambda_m,\varepsilon_m) \to (\lambda_\infty,\varepsilon_\infty)$ **it holds that** $\mu^{\varphi}_{\lambda_m,\varepsilon_m} \rightharpoonup \mu^{\varphi}_{\lambda_\infty,\varepsilon_\infty}$ **as** $m \to \infty$**.** Indeed, we have to show that for any bounded continuous function f it holds that, as $m \to \infty$ *(counting multiplicities)*

$$\int f \, \mathrm{d}\mu^{\varphi}_{\lambda_m,\varepsilon_m} = \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi(y) f(y) \to \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi(y) f(y) = \int f \, \mathrm{d}\mu^{\varphi}_{\lambda_\infty,\varepsilon_\infty} \tag{2.3}$$

Note that these are finite summations and hence well-defined. Without loss of generality we assume that $f \geq 0$. We know that the spectrum of $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ inside the support of φ is discrete, consisting of a finite number of eigenvalues, each of finite multiplicity. Let them be $\sigma_1, \dots, \sigma_M$ of respective multiplicities $N_1, \dots, N_M$. We split the proof of (2.3) into two steps.

I1. Claim:

$$\liminf_m \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_m}_{\varepsilon_m})} \varphi(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty})} \varphi(y) f(y), \qquad \textit{\textbf{(counting multiplicities)}}.$$

By the strong resolvent convergence of $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ to $\mathcal{M}^{\lambda_\infty}_{\varepsilon_\infty}$ we know that for any $\delta > 0$ small enough there are only finitely many m's for which $\mathcal{M}^{\lambda_m}_{\varepsilon_m}$ does not have at least N_i eigenvalues (counting multiplicity!) within δ of σ_i, for each $i \in \{1, \dots, M\}$. Thus, by the continuity and non-negativity of φf, $\forall \varepsilon' > 0$ and $\forall m$ large enough

$$\sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_m}^{\lambda_m})} \varphi(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi(y) f(y) - \varepsilon', \quad \textit{(counting multiplicities).}$$

This completes step **I1**.

I2. Claim:

$$\limsup_m \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_m}^{\lambda_m})} \varphi(y) f(y) \leq \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi(y) f(y), \qquad \textbf{\textit{(counting multiplicities)}}.$$

I2a. We first claim that for all but finitely many m's we have

$$\#\left(\mathrm{sp}(\mathcal{M}_{\varepsilon_m}^{\lambda_m}) \cap \operatorname{supp} \varphi\right) \leq \#\left(\mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}) \cap \operatorname{supp} \varphi\right) =: M', \quad \textit{(counting multiplicities).}$$

Indeed, suppose not. Then there would exist a subsequence (we abuse notation and still index it by m) for which $\mathcal{M}_{\varepsilon_m}^{\lambda_m}$ has (at least) $M' + 1$ distinct eigenvalues (counting multiplicity). Say $\sigma_{m,1}, \dots, \sigma_{m,M'+1}$ with respective normalised eigenfunctions $u_{m,1}, \dots, u_{m,M'+1}$. By compactness of $\operatorname{supp} \varphi$ we may pass to a subsequence (again we retain the index m) on which $\sigma_{m,i} \to \sigma_{\infty,i}$ ($\forall i \in \{1, \dots, M'+1\}$) and some $\sigma_{\infty,i} \in \operatorname{supp} \varphi$. By upper-semicontinuity (Remark 2.3i) we may pass to successive subsequences to obtain a final subsequence (still denoted m) for which additionally $u_{m,i} \to u_{\infty,i}$ strongly for each i, where $u_{\infty,i}$ is a normalised eigenfunction of $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}$ with eigenvalue $\sigma_{\infty,i}$. Moreover, as all the operators involved are self-adjoint, for each m the eigenfunctions $\{u_{m,i}\}_{i=1}^{M'+1}$ form an orthonormal system, that is: $(u_{m,i}, u_{m,j}) = \delta_{ij}$. Taking $m \to \infty$, we still have $(u_{\infty,i}, u_{\infty,j}) = \delta_{ij}$, for all $i, j \in \{1, \dots, M'+1\}$ by the bilinear continuity of the scalar product. But this implies that $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}$ has at least $M' + 1$ eigenvalues in $\operatorname{supp} \varphi$, a contradiction, proving claim **I2a**.

I2b. We can now complete the proof of **I2**. Suppose that the claimed bound fails, then there would exist $\varepsilon' > 0$ and a subsequence (still denoted m) for which

$$\sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_m}^{\lambda_m})} \varphi(y) f(y) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi(y) f(y) + \varepsilon', \qquad \textit{(counting multiplicities).}$$

for each m. Let $M_m = \#\left(\mathrm{sp}(\mathcal{M}_{\varepsilon_m}^{\lambda_m}) \cap \operatorname{supp} \varphi\right)$. Then by **I2a** we know that for all but finitely many m's we have $M_m \leq M'$. Thus some number $M'' \in \{1, \dots, M'\}$ is equal to infinitely many of the M_m's. We pass to a subsequence (still denoted m) so that $M_m = M''$ for every m. Let these eigenvalues be $\{\sigma_{m,i}\}_{i=1}^{M''}$. As in the proof of the claim above, after passing to another subsequence we have $\sigma_{m,i} \to \sigma_{\infty,i}$ for each i where $\{\sigma_{\infty,i}\}_{i=1}^{M''}$ are distinct (counting multiplicity) eigenvalues of $\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty}$. Hence, by continuity and non-negativity of $f\varphi$, we have

$$\begin{aligned} \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi(y) f(y) &\geq \sum_{i=1}^{M''} \varphi(\sigma_{\infty,i}) f(\sigma_{\infty,i}) \qquad \textit{(counting multiplicities)} \\ &= \lim_{m \to \infty} \sum_{i=1}^{M''} \varphi(\sigma_{m,i}) f(\sigma_{m,i}) \geq \sum_{y \in \mathrm{sp}(\mathcal{M}_{\varepsilon_\infty}^{\lambda_\infty})} \varphi(y) f(y) + \varepsilon' \end{aligned}$$

where the limit is on the subsequence we extracted. This is a contradiction which completes **I2**, and the weak convergence $\mu_{\lambda_m, \varepsilon_m}^{\varphi} \rightharpoonup \mu_{\lambda_\infty, \varepsilon_\infty}^{\varphi}$ follows.

II. Claim: for any $\varepsilon > 0$ and $n \in \mathbb{N}$ fixed and along any sequence $\lambda_m \to \lambda_\infty$ it holds that $\widetilde{\mu}^{\varphi}_{\lambda_m,\varepsilon,n} \rightharpoonup \widetilde{\mu}^{\varphi}_{\lambda_\infty,\varepsilon,n}$.[1] This may be shown either by the same proof as in **I**, or we may simply note that the operators involved are finite dimensional matrices whose coefficients vary continuously in λ.

III. Claim: for any fixed $\varepsilon > 0$ we have $d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda,\varepsilon,n}, \mu^{\varphi}_{\lambda,\varepsilon}) \to 0$ uniformly in $\lambda \in [0,1]$ as $n \to \infty$. We split the proof into two steps, **III1** and **III2**.

III1. Let $[0,1] \ni \lambda_n \to \lambda_\infty$. The convergence $\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n} \rightharpoonup \mu^{\varphi}_{\lambda_\infty,\varepsilon}$ follows from the same proof as in **I**. Indeed, one has to show that (counting multiplicities)

$$\int f \,\mathrm{d}\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n} = \sum_{y \in \mathrm{sp}(\widetilde{\mathcal{M}}^{\lambda_n}_{\varepsilon,n})} \varphi(y) f(y) \to \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon})} \varphi(y) f(y) = \int f \,\mathrm{d}\mu^{\varphi}_{\lambda_\infty.\varepsilon}$$

We use the fact that $\mathcal{M}^{\lambda_n}_{\varepsilon}$ has finitely many eigenvalues in $\operatorname{supp}\varphi$, so that all of them are recovered by $\widetilde{\mathcal{M}}^{\lambda_n}_{\varepsilon,n}$ for large enough n:

$$\mathrm{sp}(\widetilde{\mathcal{M}}^{\lambda_n}_{\varepsilon,n}) \cap \operatorname{supp}\varphi = \mathrm{sp}(\mathcal{M}^{\lambda_n}_{\varepsilon}) \cap \operatorname{supp}\varphi, \qquad \forall n \text{ large}, \varepsilon > 0.$$

Hence one really needs to show that (counting multiplicities)

$$\int f \,\mathrm{d}\mu^{\varphi}_{\lambda_n,\varepsilon} = \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_n}_{\varepsilon})} \varphi(y) f(y) \to \sum_{y \in \mathrm{sp}(\mathcal{M}^{\lambda_\infty}_{\varepsilon})} \varphi(y) f(y) = \int f \,\mathrm{d}\mu^{\varphi}_{\lambda_\infty,\varepsilon}.$$

This was shown in **I**.

III2. Now we are ready to prove uniform convergence. First we note that since $\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n} \rightharpoonup \mu^{\varphi}_{\lambda_\infty,\varepsilon}$, any ball around $\mu^{\varphi}_{\lambda_\infty,\varepsilon}$ within the space of finite Borel measures will contain all but finitely many of the elements of the sequence $\{\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n}\}_{n\in\mathbb{N}}$. Choosing such a ball, and omitting those elements of the sequence that do not belong to it, the metric d_{meas} makes sense there, and one can consider the distance $d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n}, \mu^{\varphi}_{\lambda_\infty,\varepsilon})$.

Uniform convergence follows from the compactness of $[0,1]$. Indeed, suppose that uniform convergence does not hold. Then $\exists\delta > 0$ such that for infinitely many n's $\exists\lambda_n \in [0,1]$ such that $d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n}, \mu^{\varphi}_{\lambda_n,\varepsilon}) > \delta$. Extract a subsequence (we abuse notation and retain the index n) for which $\lambda_n \to \lambda_\infty \in [0,1]$. From **I** we know that for all but finitely many n's we must have $d_{\mathrm{meas}}(\mu^{\varphi}_{\lambda_n,\varepsilon}, \mu^{\varphi}_{\lambda_\infty,\varepsilon}) < \delta/2$. Therefore, by the reverse triangle inequality

$$d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n}, \mu^{\varphi}_{\lambda_\infty,\varepsilon}) \geq \Big| \underbrace{d_{\mathrm{meas}}(\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n}, \mu^{\varphi}_{\lambda_n,\varepsilon})}_{>\delta} - \underbrace{d_{\mathrm{meas}}(\mu^{\varphi}_{\lambda_n,\varepsilon}, \mu^{\varphi}_{\lambda_\infty,\varepsilon})}_{<\delta/2} \Big| > \delta/2$$

for infinitely many n's, a contradiction to the weak convergence $\widetilde{\mu}^{\varphi}_{\lambda_n,\varepsilon,n} \rightharpoonup \mu^{\varphi}_{\lambda_\infty,\varepsilon}$. $\square$

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[1] These measures were defined in (2.2).

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