2. Now we turn to the denominator. We use the fact (which we have not proved yet) that $\ln(1+x) \sim x$ as $x \to 0$. Hence, $\ln(1+x^2) \sim x^2$ as $x \to 0$. Hence:
$$5\ln(1+x^2) \sim 5x^2 = o(4x), \quad x \to 0.$$
3. So we have:
$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\ln(1+x^2)} = \lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2} \underbrace{\lim_{x\to 0} \frac{\sin 2x}{2x}}_{=1} = \frac{1}{2}.$$

Important takeaway

When we want to study the limit of a complicated expression, we need to understand the asymptotic behavior of all the terms that it includes, and try to convert them to monomials.

Fundamental limits

We have the following, all as $x \to 0$: *(some of these will be proven later)*
$$\begin{aligned} \sin x &\sim x \\ 1 - \cos x &\sim \frac{1}{2}x^2 \\ \ln(1+x) &\sim x \\ e^x - 1 &\sim x \\ (1+x)^\alpha - 1 &\sim \alpha x \end{aligned}$$

6.2 Infinitesimal and infinite functions

As we have seen in the previous section, we are interested in the asymptotic study of the behavior of functions as we either approach their zeros (i.e. points where the function vanishes) or where they 'blow up' (i.e. points where they tend to $\pm\infty$). If a function $f : \mathbb{R} \to \mathbb{R}$ tends to 0 as $x \to x_0$, f is said to be **infinitesimal** at x_0. If it tends to $\pm\infty$ as $x \to x_0$, it is said to be **infinite** at x_0. *Here, as always, x_0 could be any finite value, or $\pm\infty$.*

1) If f and g are two *infinitesimal* functions at x_0, then:

- if $f = o(g)$ at x_0 then f is said to be *infinitesimal of a higher order*. Sometimes we write:
$$|f| \ll |g| \ll 1$$
to signify that $f = o(g)$ and $g = o(1)$.

- if $f \sim g$ at x_0 then f and g are said to be *infinitesimal of the same order*.

2) If f and g are two *infinite* functions at x_0, then:

- if $f = o(g)$ at x_0 then f is said to be *infinite of a lower order*. Sometimes we write:
$$1 \ll f \ll g$$
to signify that $f = o(g)$ and $\lim_{x \to x_0} f = +\infty$.
- if $f \sim g$ at x_0 then f and g are said to be *infinite of the same order*.

Ordering of important infinite functions

The following functions are ordered in terms of their infinite order as $x \to +\infty$:
(we will prove this later)
$$\log_a x \qquad x^s \qquad b^x$$
for any $a > 1,\ s > 0,\ b > 1$.

This means that:
$$\left.\begin{aligned} \log_a x &= o(x^s) \\ x^s &= o(b^x) \end{aligned}\right\} \quad \text{as } x \to +\infty$$

Ordering of important infinite sequences

The following sequences are ordered in terms of their infinite order as $n \to \infty$:
(we will prove this later)
$$\log_a n \qquad n^s \qquad b^n \qquad n! \qquad n^n$$
for any $a > 1,\ s > 0,\ b > 1$.

This means that:
$$\left.\begin{aligned} \log_a n &= o(n^s) \\ n^s &= o(b^n) \\ b^n &= o(n!) \\ n! &= o(n^n) \end{aligned}\right\} \quad \text{as } n \to \infty$$

Stirling formula

There is a precise formula for the relationship between $n!$ and n^n, known as the **Stirling formula**, it provides a way to approximate $n!$:
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \to \infty.$$
This is useful in many applications, including in statistics, where factorials appear in the binomial formula.

We don't prove it here. There is something surprising about the fact that $\sqrt{2}$, π and e all appear in this formula....

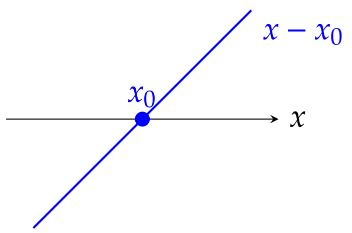
6.3 Order and principal part of infinitesimals and infinites

The most naive thing we can do, when trying to understand the behavior of an infinite or infinitesimal function, is to compare it to powers of x. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *infinite* at x_0 if $\lim_{x\to x_0} f(x) = \pm\infty$, and it is *infinitesimal* at x_0 if $\lim_{x\to x_0} f(x) = 0$.

Infinitesimal functions at a finite point x_0

Suppose that $x_0 \in \mathbb{R}$, and that $\lim_{x\to x_0} f(x) = 0$. Then

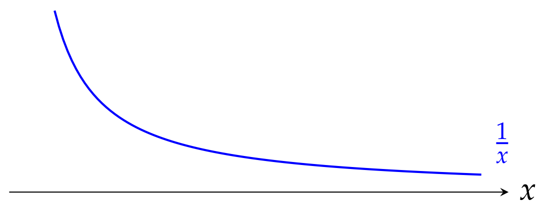
$$\textit{we want to compare } f \textit{ to powers of } \varphi(x) = x - x_0$$



Infinitesimal functions at an infinite point x_0

Suppose that $x_0 \in \{\pm\infty\}$, and that $\lim_{x\to x_0} f(x) = 0$. Then

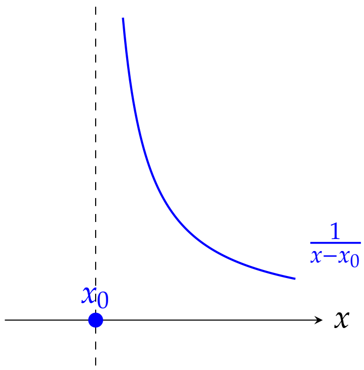
$$\textit{we want to compare } f \textit{ to powers of } \varphi(x) = \frac{1}{x}$$



Infinite functions at a finite point x_0

Suppose that $x_0 \in \mathbb{R}$, and that $\lim_{x\to x_0} f(x) = \pm\infty$. Then

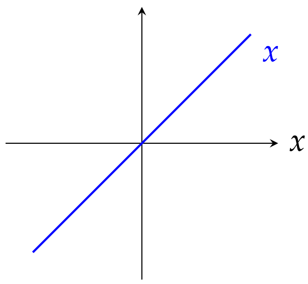
$$\textit{we want to compare } f \textit{ to powers of } \varphi(x) = \frac{1}{x - x_0}$$



Infinite functions at an infinite point x_0

Suppose that $x_0 \in \{\pm\infty\}$, and that $\lim_{x\to x_0} f(x) = \pm\infty$. Then

we want to compare f to powers of $\varphi(x) = x$



In all the above cases, sometimes we may need φ to be non-negative. So we may need to introduce an absolute value in the definitions of the various φ.

Order of an infinitesimal/infinite function

Let f be infinitesimal or infinite at x_0. If there exists $\alpha > 0$ such that

$$f \asymp \varphi^\alpha, \quad x \to x_0$$

then we say that α is the **order** of f at x_0 with respect to φ.

Observe that if f has an order it is *unique*. Furthermore, by definition of the symbol $\asymp$, $f \asymp \varphi^\alpha$, $x \to x_0$, means that

$$\lim_{x\to x_0} \frac{f(x)}{\varphi^\alpha(x)} = \ell \in \mathbb{R} \setminus \{0\}.$$

This can also be written as:

$$f \sim \ell\varphi^\alpha, \quad x \to x_0$$

which can be rewritten as

$$f = \ell\varphi^\alpha + o(\varphi^\alpha), \quad x \to x_0.$$

Principal part of an infinitesimal/infinite function

If $f = \ell\varphi^\alpha + o(\varphi^\alpha)$ as $x \to x_0$ then

$$p(x) = \ell\varphi^\alpha(x)$$

is called the principal part of f at x_0 with respect to φ.

Examples

Example 6.3: Consider the function

$$f(x) = \sin x - \tan x, \qquad \text{near } x_0 = 0.$$

Since $\sin 0 = \tan 0 = 0$, f is infinitesimal at $x = 0$. We can write:

$$\begin{aligned}
\sin x - \tan x &= \sin x - \frac{\sin x}{\cos x} \\
&= \frac{\sin x \cdot (\cos x - 1)}{\cos x} \\
&\sim \frac{x \cdot (-\frac{1}{2}x^2)}{1} \\
&= -\frac{1}{2}x^3, \qquad x \to 0.
\end{aligned}$$

Hence f is infinitesimal of order 3 at $x = 0$ with respect to $\varphi(x) = x$. The principal part is $p(x) = -\frac{1}{2}x^3$ and we can write:

$$\sin x - \tan x = -\frac{1}{2}x^3 + o(x^3), \quad x \to 0.$$

Example 6.4: Consider the function

$$f(x) = \sqrt{x^2+3} - \sqrt{x^2-1}.$$

As $x \to +\infty$, $f(x)$ tends to 0 (verify that you can see why!). So f is infinitesimal at $x_0 = +\infty$. We'll want to compare it to $\varphi(x) = \frac{1}{x}$. First we want the roots in the denominator, so we multiply and divide by $\sqrt{x^2+3} + \sqrt{x^2-1}$ to get:

$$\begin{aligned}
f(x) &= \sqrt{x^2+3} - \sqrt{x^2-1} \\
&= \frac{(x^2+3) - (x^2-1)}{\sqrt{x^2+3} + \sqrt{x^2-1}} \\
&= \frac{1}{x} \cdot \frac{4}{\sqrt{1+\frac{3}{x^2}} + \sqrt{1-\frac{1}{x^2}}}
\end{aligned}$$

To compare to $\varphi(x) = \frac{1}{x}$ we need to look at

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi^\alpha(x)}$$

and try to identify the correct α. We see that we get:

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi^\alpha(x)} = \lim_{x\to+\infty} \frac{x^\alpha}{x} \cdot \frac{4}{\sqrt{1+\frac{3}{x^2}} + \sqrt{1-\frac{1}{x^2}}}$$

so that with $\alpha = 1$ we have

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi^\alpha(x)} = \lim_{x\to+\infty} \frac{4}{\sqrt{1+\frac{3}{x^2}} + \sqrt{1-\frac{1}{x^2}}} = 2.$$

Hence f is infinitesimal of order 1 at $+\infty$ with respect to $\varphi(x) = x^{-1}$. The principal part is $p(x) = 2x^{-1}$ and we can write

$$\sqrt{x^2+3} - \sqrt{x^2-1} = 2x^{-1} + o(x^{-1}), \qquad x \to +\infty.$$

Example 6.5: Consider the function

$$f(x) = \sqrt{9x^5 + 7x^3 - 1}$$

which is infinite as $x \to +\infty$. To determine the order we compare it with $\varphi(x) = x$:

$$\begin{aligned}\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} &= \lim_{x\to+\infty} \frac{\sqrt{9x^5+7x^3-1}}{x^\alpha} \\ &= \lim_{x\to+\infty} \frac{x^{\frac{5}{2}}\sqrt{9+7x^{-2}-x^{-5}}}{x^\alpha}.\end{aligned}$$

This suggests choosing $\alpha = \frac{5}{2}$. Then we have:

$$\lim_{x\to+\infty} \frac{f(x)}{x^{\frac{5}{2}}} = \lim_{x\to+\infty} \sqrt{9+7x^{-2}-x^{-5}} = 3.$$

Hence f is infinite of order $\frac{5}{2}$ at $+\infty$ with respect to $\varphi(x) = x$. The principal part is $p(x) = 3x^{\frac{5}{2}}$ and we can write

$$\sqrt{9x^5+7x^3-1} = 3x^{\frac{5}{2}} + o(x^{\frac{5}{2}}), \qquad x \to +\infty.$$

6.4 Asymptotes

We have already seen horizontal and vertical asymptotes. However it is possible to have slanted asymptotes. We say that a function $f(x)$ **behaves asymptotically as $x \to +\infty$ like the affine function** $y = ax + b$ if

$$\lim_{x\to+\infty} (f(x) - (ax+b)) = 0.$$