Chapter 6

Local comparison of functions

In this chapter we shall gather some tools that will allow us to study the *asymptotic* behavior of functions. The asymptotic behavior of a function $f(x)$ can refer either to its behavior as $x \to \pm\infty$, or as $x \to x_0 \in \mathbb{R}$, where $f(x)$ might tend to 0 or to $\pm\infty$.

6.1 Landau symbols

Let $f, g : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Assume that the limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell$$

exists (can be finite or infinite). We introduce the **Landau symbols**:

Big O

If ℓ is finite, then we say that f **is controlled by** g as $x \to x_0$, and we write

$$f = O(g), \quad x \to x_0.$$

We often say that f **is big O of** g as $x \to x_0$.

Same order

If ℓ is finite and $\ell \neq 0$, then we say that f **has the same order of magnitude as** g as $x \to x_0$, and we write

$$f \asymp g, \quad x \to x_0.$$

Asymptotically equivalent

If $\ell = 1$, then we say that f **is equivalent to** g as $x \to x_0$, and we write

$$f \sim g, \quad x \to x_0.$$

Little o

If $\ell = 0$, then we say that f **is negligible with respect to** g as $x \to x_0$, and we write

$$f = o(g), \quad x \to x_0.$$

We often say that f **is little** o **of** g as $x \to x_0$.

$\ell = \pm\infty$

If $\ell = \pm\infty$, we need to look at $\frac{g}{f}$ instead!

Example 6.1: 1. Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ we have

$$\sin x \sim x, \quad x \to 0.$$

2. Since $\lim_{x\to +\infty} \frac{\sin x}{x} = 0$ we have

$$\sin x = o(x), \quad x \to +\infty.$$

3. Since $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ we have

$$1 - \cos x \asymp x^2, \quad x \to 0.$$

Properties of Landau symbols

1. Observe that f and g having the same order, or being asymptotically equivalent, or being little o, are all subcases of being Big O, that is:

$$\begin{aligned} f \asymp g &\quad\Rightarrow\quad f = O(g) \\ f \sim g &\quad\Rightarrow\quad f = O(g) \\ f = o(g) &\quad\Rightarrow\quad f = O(g) \end{aligned}$$

   all as $x \to x_0$. Another implication is:

$$f \sim g \quad\Rightarrow\quad f \asymp g$$

   as $x \to x_0$. Conversely:

$$f \asymp g \quad\Rightarrow\quad f \sim \ell g$$

   as $x \to x_0$.

2.
$$f \sim g \quad\Leftrightarrow\quad f = g + o(g)$$

   as $x \to x_0$.

3.

$$\begin{aligned} f = \mathcal{O}(g) \quad &\Leftrightarrow \quad f = \mathcal{O}(\lambda g),\ \forall \lambda \neq 0 \\ f = o(g) \quad &\Leftrightarrow \quad f = o(\lambda g),\ \forall \lambda \neq 0 \end{aligned}$$

all as $x \to x_0$.

4.

$$\begin{aligned} f = \mathcal{O}(1),\ x \to x_0 \quad &\Leftrightarrow \quad f(x) \xrightarrow{x \to x_0} \ell \in \mathbb{R} \\ f = o(1),\ x \to x_0 \quad &\Leftrightarrow \quad f(x) \xrightarrow{x \to x_0} 0 \end{aligned}$$

In particular, in both cases, f is bounded in a neighborhood of x_0.

5.

$$f \text{ is continuous at } x_0 \quad \Leftrightarrow \quad f(x) = f(x_0) + o(1), \text{ as } x \to x_0.$$

Monomials

One of the simplest functions is the monomial power of x: x^n. We therefore want to be able to compare them near where they vanish ($x_0 = 0$) and near where they tend to infinity ($x_0 = \pm\infty$).

Near 0

Near $x_0 = 0$ we have:

$$x^n = o(x^m),\ x \to 0, \quad \Leftrightarrow \quad n > m,$$

since $x^{n-m} \to 0$ as $x \to 0$ when $n > m$. This implies that **near 0, bigger powers are negligible**.

Near $\pm\infty$

Near $x_0 = \pm\infty$ we have:

$$x^n = o(x^m),\ x \to \pm\infty, \quad \Leftrightarrow \quad n < m,$$

since $x^{n-m} = \frac{1}{x^{m-n}} \to 0$ as $x \to \pm\infty$ when $n < m$. This implies that **near $\pm\infty$, smaller powers are negligible**.

Further properties of Landau symbols

Proposition 6.1: Consider functions $f, \tilde{f}, g, \tilde{g} : \mathbb{R} \to \mathbb{R}$ such that $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $x \to x_0$. Then

$$\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \tilde{f}(x)\tilde{g}(x)$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\tilde{f}(x)}{\tilde{g}(x)}$$

Proof. Let us prove the first claim. We multiply and divide by the *tilde* functions as follows

$$\begin{aligned}\lim_{x\to x_0} f(x)g(x) &= \lim_{x\to x_0} \frac{f(x)}{\tilde{f}(x)}\frac{g(x)}{\tilde{g}(x)}\tilde{f}(x)\tilde{g}(x)\\ &= \underbrace{\left(\lim_{x\to x_0}\frac{f(x)}{\tilde{f}(x)}\right)}_{=1}\underbrace{\left(\lim_{x\to x_0}\frac{g(x)}{\tilde{g}(x)}\right)}_{=1}\left(\lim_{x\to x_0}\tilde{f}(x)\tilde{g}(x)\right)\end{aligned}$$

and the proof is complete. *Note that we have used the fact that we obtained the product of functions that in the limit give meaningful expressions (otherwise we would not have been allowed to take limits of the individual parts of the product).* □

Corollary 6.2: Consider functions $f, f_1, g, g_1 : \mathbb{R} \to \mathbb{R}$ and assume that $f_1 = o(f)$ and $g_1 = o(g)$ both as $x \to x_0$. Then

$$\lim_{x\to x_0}(f(x) \pm f_1(x)(g(x) \pm g_1(x)) = \lim_{x\to x_0} f(x)g(x)$$

$$\lim_{x\to x_0}\frac{f(x) \pm f_1(x)}{g(x) \pm g_1(x)} = \lim_{x\to x_0}\frac{f(x)}{g(x)}$$

Corollary 6.3: Consider functions $f, \tilde{f}, g, \tilde{g} : \mathbb{R} \to \mathbb{R}$ such that $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $x \to x_0$. Then

$$\begin{array}{ccccccc} f = O(g) & \Leftrightarrow & f = O(\tilde{g}) & \Leftrightarrow & \tilde{f} = O(g) & \Leftrightarrow & \tilde{f} = O(\tilde{g}) \\ f = o(g) & \Leftrightarrow & f = o(\tilde{g}) & \Leftrightarrow & \tilde{f} = o(g) & \Leftrightarrow & \tilde{f} = o(\tilde{g}) \end{array}$$

all as $x \to x_0$.

Warning!

These rules do not apply to sums and differences. For instance, consider

$$\begin{aligned} f(x) &= x & \tilde{f}(x) &= x + 1\\ g(x) &= x & \tilde{g}(x) &= x \end{aligned}$$

Then $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $x \to +\infty$. However,

$$\lim_{x\to+\infty}(f(x) - g(x)) = \lim_{x\to+\infty} 0 = 0 \neq 1 = \lim_{x\to+\infty} 1 = \lim_{x\to+\infty}(\tilde{f}(x) - \tilde{g}(x)).$$

Example 6.2: Compute

$$\lim_{x\to 0}\frac{\sin 2x + x^3}{4x + 5\ln(1 + x^2)}.$$

1. First we simplify the numerator. We know that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. This implies that $\sin 2x \sim 2x$ as $x \to 0$. Hence (using Corollary 6.3):

$$x^3 = o(2x) = o(\sin 2x), \quad x \to 0.$$

2. Now we turn to the denominator. We use the fact (which we have not proved yet) that $\ln(1+x) \sim x$ as $x \to 0$. Hence, $\ln(1+x^2) \sim x^2$ as $x \to 0$. Hence:

$$5\ln(1+x^2) \sim 5x^2 = o(4x), \quad x \to 0.$$

3. So we have:

$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\ln(1+x^2)} = \lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2} \underbrace{\lim_{x\to 0} \frac{\sin 2x}{2x}}_{=1} = \frac{1}{2}.$$

Important takeaway

When we want to study the limit of a complicated expression, we need to understand the asymptotic behavior of all the terms that it includes, and try to convert them to monomials.