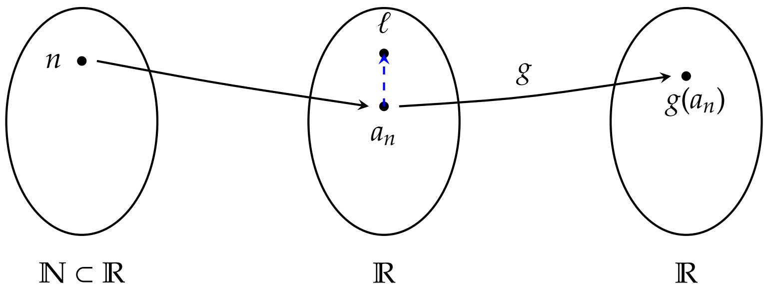
Nonexistence of a limit

To show that a limit $\lim_{x\to x_0} g(y)$ *doesn't* exist we can rely on the previous results. A common method is as follows: as in the figure below, compose g with a sequence a_n such that $\lim_{n\to\infty} a_n = \ell$. Then try to find another sequence, $\{b_n\}_{n\in\mathbb{N}}$, also satisfying $\lim_{n\to\infty} b_n = \ell$, but for which $\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n)$. Then g doesn't have a limit at ℓ.



Theorem 5.14 (Criterion for nonexistence of a limit)**:** Let $g : \mathbb{R} \to \mathbb{R}$ be defined in a neighborhood of $\ell \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ (possibly excluding ℓ itself). Suppose that there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = \ell = \lim_{n\to\infty} b_n$ and such that

$$\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n).$$

Then $g(y)$ does not have a limit as $y \to \ell$.

Proof. By contradiction. If the limit existed, then the Substitution Theorem would imply that

$$\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y) = \lim_{n\to\infty} g(b_n),$$

in contradiction to the assumption. □

5.6 Theorems on limits of sequences

We can now continue the analysis of sequences, which we begun in Section 4.2. To simplify the presentation, let us agree that we say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfies a property **for all large** n if there exists $N \in \mathbb{N}$ such that for all $n > N$ the sequence satisfies this property. The results we obtained for functions all carry over to sequences, so we can state the following 'big' theorem:

Theorem 5.15:

1. The limit of a sequence (if exists) is unique.
2. A convergent sequence is bounded.
3. A sequence that is monotone for all large n cannot be indeterminate.
4. For sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$, if $a_n \leq b_n$ for all large n, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.
5. For sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$, if for all large n, $a_n \leq b_n \leq c_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then b_n has a limit and it is the same limit.

6. If two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ have limits $\lim_{n\to\infty} a_n = \ell_a$ and $\lim_{n\to\infty} b_n = \ell_b$, then
   (a) $\lim_{n\to\infty}(a_n \pm b_n) = \ell_a \pm \ell_b$
   (b) $\lim_{n\to\infty}(a_n \cdot b_n) = \ell_a \cdot \ell_b$
   (c) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\ell_a}{\ell_b}$, if for all large n, $b_n \neq 0$,

   whenever the expressions on the right hand side are meaningful.
7. If $\{a_n\}_{n\in\mathbb{N}}$ has the limit ℓ and $g : \mathbb{R} \to \mathbb{R}$ is defined in a neighborhood of ℓ, then
   (a) if $\ell \in \mathbb{R}$ and g is continuous at ℓ, then $\lim_{n\to\infty} g(a_n) = g(\ell)$,
   (b) if $\ell = \pm\infty$ and $\lim_{y\to\ell} g(y)$ exists, then $\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y)$.
8. $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.
9. If a sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded, and another sequence $\{b_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty}(a_n b_n) = 0$.

Proof. The proof is completely analogous to the various proofs we've seen for functions. We skip it here. □

For sequences there is an additional useful tool, which relies in the fact that sequences are discrete (as opposed to functions on $\mathbb{R}$):

Theorem 5.16 (Ratio Test): Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence that for is positive for all large n (i.e. there exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n > 0$). Assume that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ exists (it may be finite or infinite). Then

- if $q < 1$, then $\lim_{n\to\infty} a_n = 0$,
- if $q > 1$, then $\lim_{n\to\infty} a_n = +\infty$,
- if $q = 1$, it is impossible to determine whether or not the sequence has a limit.

Note that the theorem applies also for sequences that are negative for all large n.

Proof. The proof is simple and we skip it here. □

Example 5.16: Consider a sequence that we have previously seen:

$$a_n = \frac{n!}{n^{100}}.$$

We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{100}}}{\frac{n!}{n^{100}}} = \frac{n^{100}(n+1)!}{(n+1)^{100}n!} = \underbrace{\left(\frac{n}{n+1}\right)^{100}}_{\to 1}\underbrace{(n+1)}_{\to+\infty},$$

hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$$

so that the sequence a_n diverges.

5.7 Fundamental limits and indeterminate forms of exponential type

We have seen before that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Now we show the same result for the function $\left(1+\frac{1}{x}\right)^x$:

Claim: The function $\left(1+\frac{1}{x}\right)^x$ has limits as $x \to \pm\infty$, and

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Proof. Observe that the function $\left(1+\frac{1}{x}\right)^x$ is defined when $1+\frac{1}{x} > 0$ and $x \neq 0$. Hence it is defined when either $x > 0$ or $x < -1$. We prove for the case $x \to +\infty$.

Let $n = \lceil x \rceil$. Then

$$n \leq x < n+1.$$

Hence

$$\frac{1}{n+1} < \frac{1}{x} \leq \frac{1}{n}$$

$$\Downarrow$$

$$1+\frac{1}{n+1} < 1+\frac{1}{x} \leq 1+\frac{1}{n}$$

$$\Downarrow$$

$$\left(1+\frac{1}{n+1}\right)^n \leq \left(1+\frac{1}{n+1}\right)^x < \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{n}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}$$

So we have:

$$\underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{\to e}\underbrace{\left(1+\frac{1}{n+1}\right)^{-1}}_{\to 1} < \left(1+\frac{1}{x}\right)^x < \underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}\underbrace{\left(1+\frac{1}{n}\right)}_{\to 1}$$

So by the Squeeze Theorem,

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e.$$

The proof in the case $x \to -\infty$ follows similarly, taking caution with signs. □

Observe that by substituting $y = \frac{1}{x}$ we have:

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to 0}(1+y)^{\frac{1}{y}} = e.$$

Claim:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$

Proof.
$$\frac{\ln(1+x)}{x} = \frac{1}{x}\ln(1+x) = \ln\left((1+x)^{\frac{1}{x}}\right).$$
Hence, using the continuity of the logarithm, we have:
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) = e$$
where in the last equality we have used the previous remark (above). □

Claim:
$$\lim_{x\to 0}\frac{e^x-1}{x} = 1.$$

Proof. Follows from the previous claim with an appropriate substitutions, we skip this here. □

Useful identities

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$
$$\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a, \qquad (a\in\mathbb{R})$$
$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \frac{1}{\ln a}, \qquad (a>0,\ a\neq 1)$$
$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a, \qquad (a>0)$$
$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x} = \alpha, \qquad (\alpha\in\mathbb{R})$$

Power functions

Limits of powers of functions

Let $h(x) = [f(x)]^{g(x)}$, let $x_0 \in \{-\infty\}\cup\mathbb{R}\cup\{+\infty\}$ and assume that f, g have limits as $x \to x_0$ and that $f > 0$ near x_0. Observe that $h = e^{g\ln f}$. Hence, by the continuity of the exponential and the fact that for continuous functions we can commute the operations of taking the limit and applying the function:
$$\lim_{x\to x_0}\left([f(x)]^{g(x)}\right) = e^{\lim_{x\to x_0}(g(x)\ln f(x))}.$$

So we need to study the exponential of $\lim_{x\to x_0}(g(x)\ln f(x))$. This is the limit of the product of two functions. We know that it is problematic if we get $0\cdot\infty$. Hence we need to investigate thoroughly in these cases:

1. $\lim_{x\to x_0} g(x) = \pm\infty$ and $\lim_{x\to x_0} f(x) = 1$, so that we get 1^∞.
2. $\lim_{x\to x_0} g(x) = 0$ and $\lim_{x\to x_0} f(x) = 0$, so that we get 0^0.
3. $\lim_{x\to x_0} g(x) = 0$ and $\lim_{x\to x_0} f(x) = +\infty$, so that we get ∞^0.

Example 5.17: Determine $\lim_{x\to+\infty} x^{\frac{1}{x}}$.

We see that this has the form ∞^0. Let $y = \frac{1}{x}$, so that the problem becomes $\lim_{y\to 0^+}(1/y)^y$. We see that

$$x^{\frac{1}{x}} = \left(\frac{1}{y}\right)^y = e^{y\ln\frac{1}{y}} = e^{-y\ln y}.$$

We will later prove that $\lim_{y\to 0^+}(y\ln y) = 0$, so that

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = \lim_{y\to 0^+} e^{-y\ln y} = e^{\lim_{y\to 0+}(-y\ln y)} = e^0 = 1.$$