Corollary 5.10 (Squeeze to 0 Theorem): Let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded in a neighborhood of x_0. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy $\lim_{x\to x_0} g(x) = 0$. Then

$$\lim_{x\to x_0} f(x) \cdot g(x) = 0.$$

Proof. Observe that $\lim_{x\to x_0} g(x) = 0$ is satisfied if and only if $\lim_{x\to x_0} |g(x)| = 0$. By assumption, f is bounded on a neighborhood of x_0. This means that there exists $M > 0$ such that $|f(x)| < M$ for all x in this neighborhood. Hence, on this neighborhood of x_0 we have

$$0 \le |f(x) \cdot g(x)| \le Mg(x)$$

and the claim follows from Theorem 5.9. □

Theorem 5.11 (Squeeze to $\pm\infty$ Theorem): Let $f, g : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\}\cup\mathbb{R}\cup\{+\infty\}$. If

$$\lim_{x\to x_0} f(x) = +\infty \qquad \text{and} \qquad f \le g \text{ in a neighborhood of } x_0$$
$$\textit{(excluding } x_0 \textit{ itself)}$$

then

$$\lim_{x\to x_0} g(x) = +\infty.$$

An analogous statement (with obvious modifications) can be made for the case when the limit is $-\infty$.

Proof. The proof is a simple adaptation of previous proofs, we skip it here. □

Example 5.8: Show that

$$\lim_{x\to -\infty} \left(x^2 - e^x - 3\sin x - 8\right) = +\infty.$$

Observe that for $x < 0$ we have:

$$x^2 - e^x - 3\sin x - 8 \ge x^2 - 1 - 3 - 8 = x^2 - 12.$$

Letting $f(x) = x^2 - 12$, we see that $\lim_{x\to-\infty} f(x) = +\infty$, and the theorem implies the required result.

5.4 Indeterminate forms of algebraic type

We go back to the meaningless expressions:

Meaningless expressions

$$+\infty - \infty \qquad -\infty + \infty \qquad \pm\infty \cdot 0 \qquad \frac{\pm\infty}{\pm\infty} \qquad \frac{0}{0}$$

Here we want to show that for algebraic functions (i.e. polynomials, rational functions or functions involving roots of polynomials) we can sometimes make sense of such expressions, by careful inspection and simple manipulation.

Simple examples

Example 5.9: The indeterminate form $+\infty - \infty$ can yield any result:

- $\lim_{x\to+\infty} ((x+1)-x) = 1$
- $\lim_{x\to+\infty} (x-(x+1)) = -1$
- $\lim_{x\to+\infty} (2x-x) = +\infty$
- $\lim_{x\to+\infty} (x-2x) = -\infty$
- $\lim_{x\to+\infty} ((x+\sin x)-x)$ does not exist (oscillates)

Example 5.10: The indeterminate form $+\infty \cdot 0$ can yield any result:

- $\lim_{x\to+\infty} \left(x \cdot \frac{1}{x}\right) = 1$
- $\lim_{x\to+\infty} \left(x \cdot \frac{1}{x^2}\right) = 0$
- $\lim_{x\to+\infty} \left(x^2 \cdot \frac{1}{x}\right) = +\infty$
- $\lim_{x\to+\infty} \left(x \cdot \frac{\sin x}{x}\right)$ does not exist (oscillates)

Example 5.11: The indeterminate form $\frac{+\infty}{+\infty}$ can yield any result:

- $\lim_{x\to+\infty} \left(\frac{x}{x}\right) = 1$
- $\lim_{x\to+\infty} \left(\frac{x}{x^2}\right) = 0$
- $\lim_{x\to+\infty} \left(\frac{x^2}{x}\right) = +\infty$
- $\lim_{x\to+\infty} \left(\frac{x+x\sin x}{x}\right)$ does not exist (oscillates between 0 and 2)

Example 5.12: The indeterminate form $\frac{0}{0}$ can yield any result:

- $\lim_{x\to0} \left(\frac{x}{x}\right) = 1$
- $\lim_{x\to0} \left(\frac{x^2}{x}\right) = 0$
- $\lim_{x\to0} \left(\frac{x}{x^2}\right) = +\infty$
- $\lim_{x\to0} \left(\frac{\sin x}{x}\right) = 1$
- $\lim_{x\to0} \left(\frac{x\sin(1/x)}{x}\right)$ does not exist (oscillates)

Polynomials

Consider the following polynomial (assume $a_n \neq 0$):

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

It can be rewritten as

$$p(x) = x^n \underbrace{\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)}_{\text{converges to } a_n \text{ as } x \to \pm\infty}.$$

Since we know how the part in the brackets behaves, we can deduce that:

$$\lim_{x\to+\infty} p(x) = \begin{cases} +\infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

and

$$\lim_{x\to-\infty} p(x) = \begin{cases} +\infty & \text{if } a_n > 0 \text{ and } n \text{ is even} \\ -\infty & \text{if } a_n > 0 \text{ and } n \text{ is odd} \\ +\infty & \text{if } a_n < 0 \text{ and } n \text{ is odd} \\ -\infty & \text{if } a_n < 0 \text{ and } n \text{ is even} \end{cases}$$

Rational functions

Consider the following rational function (assume $a_n \neq 0$, $b_m \neq 0$):

$$\begin{aligned} r(x) = \frac{p(x)}{q(x)} &= \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} \\ &= x^{n-m} \underbrace{\frac{a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n}}{b_m + b_{m-1}x^{-1} + \cdots + b_1 x^{-m+1} + b_0 x^{-m}}}_{\text{converges to } \frac{a_n}{b_m} \text{ as } x\to\pm\infty} \end{aligned}$$

We therefore have

$$\lim_{x\to\pm\infty} r(x) = \begin{cases} \infty & \text{if } n > m \\ \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

The first case ($n > m$) requires further analysis (as in the case of a polynomial) to determine the type of infinity (i.e. whether the limit is $+\infty$ or $-\infty$).

Other algebraic functions

If we encounter a problem with roots, our first goal is to get rid of these roots, at least in the numerator. This can often be achieved by using the fact that $(a+b)(a-b) = a^2 - b^2$:

$$\frac{\sqrt{f(x)} + \sqrt{g(x)}}{h(x)} = \frac{\sqrt{f(x)} + \sqrt{g(x)}}{h(x)} \cdot \frac{\sqrt{f(x)} - \sqrt{g(x)}}{\sqrt{f(x)} - \sqrt{g(x)}} = \frac{f(x) - g(x)}{h(x)(\sqrt{f(x)} - \sqrt{g(x)})}.$$

We have thereby gotten rid of the roots in the numerator, and, hopefully, the resulting expression is easier to deal with.

Examples

Example 5.13: Compute
$$\lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1-2x}}{3x}.$$

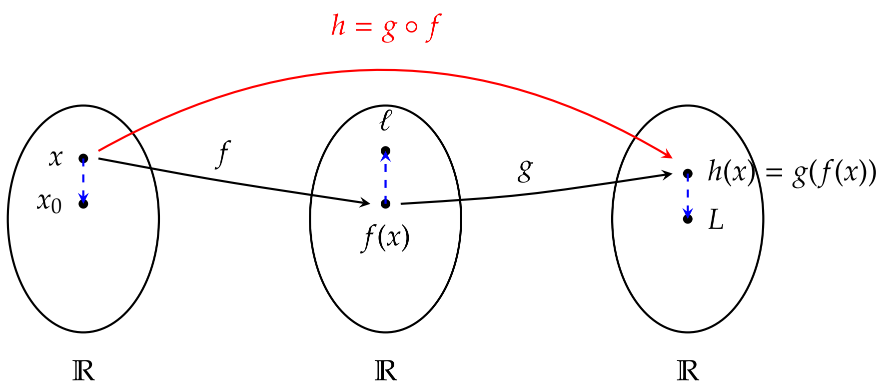
Observe that, in the limit, we get an expression of the form $\frac{0}{0}$, so we cannot determine the limit. To proceed, we multiply numerator and denominator by $\sqrt{1+5x}+\sqrt{1-2x}$ to get:

$$\begin{aligned}
\frac{\sqrt{1+5x}-\sqrt{1-2x}}{3x} &= \frac{\sqrt{1+5x}-\sqrt{1-2x}}{3x}\cdot\frac{\sqrt{1+5x}+\sqrt{1-2x}}{\sqrt{1+5x}+\sqrt{1-2x}}\\
&= \frac{1+5x-(1-2x)}{3x\left(\sqrt{1+5x}+\sqrt{1-2x}\right)}\\
&= \frac{7x}{3x\left(\sqrt{1+5x}+\sqrt{1-2x}\right)}\\
&= \frac{7}{3}\cdot\underbrace{\frac{1}{\sqrt{1+5x}+\sqrt{1-2x}}}_{\text{this tends to } \frac{1}{2} \text{ as } x\to 0} = \frac{7}{6}.
\end{aligned}$$

We have therefore found that:
$$\lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1-2x}}{3x} = \frac{7}{6}.$$

5.5 Substitution Theorem

We now want to understand how limits behave under composition of functions: if we have $f, g : \mathbb{R} \to \mathbb{R}$, with $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{y\to \ell} g(y) = L$, then we want to conclude that $\lim_{x\to x_0} g(f(x)) = L$.



This is indeed true:

Theorem 5.12 (Substitution Theorem): Let $f, g : \mathbb{R} \to \mathbb{R}$ and let $x_0, \ell \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Suppose that
$$\lim_{x\to x_0} f(x) = \ell$$

and that g is defined on a neighborhood of ℓ (possibly excluding ℓ itself), satisfying:

- if $\ell \in \mathbb{R}$ then g is continuous at ℓ, and
- if $\ell = \pm\infty$ then $\lim_{y\to\ell} g(y)$ exists (possibly infinite).

Then $h = g \circ f$ has a limit as $x \to x_0$ and:

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to\ell} g(y).$$

Proof. We skip the proof. It is straightforward, and can be found in the book. □

Composition of continuous functions

Observe that if $\ell \in \mathbb{R}$ and g is continuous at ℓ, then $\lim_{y\to\ell} g(y) = g(\ell)$ so that the conclusion of the theorem simplifies to:

$$\lim_{x\to x_0} g(f(x)) = g(\ell) = g(\lim_{x\to x_0} f(x)).$$

That is, in this case, the operation of applying the function g and the operation of taking the limit $x \to x_0$ commute (the order at which we take them can be replaced).

Corollary 5.13: If f is continuous at x_0 and g is continuous at ℓ then $g \circ f$ is continuous at x_0.

Proof. This is immediate from the last comment. Denote $h = g \circ f$. Then:

$$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(f(x)) = g(\lim_{x\to x_0} f(x)) = g(f(x_0)) = h(x_0).$$

□

Example 5.14: Compute

$$\lim_{x\to 0} \frac{\sin x^2}{x^2}.$$

We see that this function is the composition of $f(x) = x^2$ with $g(y) = \frac{\sin y}{y}$, for $y \neq 0$. We know that $\lim_{y\to 0} g(y) = 1$, so we complete g by defining $g(0) = 1$. Now f and g are continuous on $\mathbb{R}$. Using the fact that $\lim_{x\to 0} f(x) = 0$ we have

$$\lim_{x\to 0} \frac{\sin x^2}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

Example 5.15: Compute

$$\lim_{x\to +\infty} \ln\left(\sin\left(\frac{1}{x}\right)\right).$$

Here we have

$$f(x) = \sin\left(\frac{1}{x}\right) \qquad \text{and} \qquad g(y) = \ln y.$$

As $x \to +\infty$, $\frac{1}{x} \to 0$. We know that $\sin 0 = 0$. The logarithm $\ln y$ isn't defined for $y \leq 0$, however we know that $\lim_{y\to 0^+} \ln y = -\infty$. So we have:

$$\lim_{x\to +\infty} \ln\left(\sin\left(\frac{1}{x}\right)\right) = \lim_{y\to 0^+} \ln y = -\infty.$$