5.2 Algebra of limits

Given some functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, our goal is to understand how limits of $f \pm g$ might behave, as well as limits of $f \cdot g$ and $\frac{f}{g}$. In general we should exercise caution!

Example 5.1: Let $C \in \mathbb{R}$. Consider the sequences $a_n = n + C$ and $b_n = -n$. Then:

$$\lim_{n\to\infty} a_n = +\infty \qquad \lim_{n\to\infty} b_n = -\infty$$

and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (n + C - n) = \lim_{n\to\infty} C = C.$$

This suggests that $+\infty - \infty = C$ for any C. The analogous example for functions would involve $f(x) = x + C$ and $g(x) = -x$.

Example 5.2: Consider the sequences $a_n = 2n$ and $b_n = -n$. Then:

$$\lim_{n\to\infty} a_n = +\infty \qquad \lim_{n\to\infty} b_n = -\infty$$

and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (2n - n) = \lim_{n\to\infty} n = +\infty.$$

Example 5.3: Consider the sequences $a_n = n$ and $b_n = -2n$. Then:

$$\lim_{n\to\infty} a_n = +\infty \qquad \lim_{n\to\infty} b_n = -\infty$$

and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (n - 2n) = \lim_{n\to\infty} (-n) = -\infty.$$

These examples demonstrate that we should be cautious when taking limits of sums or differences of sequences and functions. In particular, these examples show that $+\infty - \infty$ is a meaningless expression. Similar absurdities can be achieved by multiplication or division of sequences or functions. Let's list the meaningless expressions:

Meaningless expressions

The following expressions do not make sense:

$$+\infty - \infty \qquad -\infty + \infty \qquad \pm\infty \cdot 0 \qquad \frac{\pm\infty}{\pm\infty} \qquad \frac{0}{0}$$

In contrast, here are the meaningful expressions:

Meaningful expressions

The following expressions *do* make sense:

$$\begin{aligned}
+\infty + C &= +\infty \quad (\text{if } C \in \mathbb{R} \cup \{+\infty\}) \\
-\infty + C &= -\infty \quad (\text{if } C \in \{-\infty\} \cup \mathbb{R}) \\
\pm\infty \cdot C &= \pm\infty \quad (\text{if } C \in \mathbb{R}_+ \cup \{+\infty\}) \\
\pm\infty \cdot C &= \mp\infty \quad (\text{if } C \in \{-\infty\} \cup \mathbb{R}_-) \\
\frac{\pm\infty}{C} &= \pm\infty \quad (\text{if } C \in \mathbb{R}_+) \\
\frac{\pm\infty}{C} &= \mp\infty \quad (\text{if } C \in \mathbb{R}_-) \\
\frac{C}{\pm\infty} &= 0 \quad (\text{if } C \in \mathbb{R})
\end{aligned}$$

A more delicate meaningful expression is

$$\frac{C}{0} = \infty \quad (\text{if } C \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \setminus \{0\})$$

However to determine whether it is $+\infty$ or $-\infty$ we need to look at the numerator and the denominator in a neighborhood of the point in question. This could lead to different left and right limits (in the case that the denominator changes sign at the point in question).

Example 5.4: We start with a simple example:

$$f(x) = \frac{1}{x}.$$

At $x = 0$, we have an expression of the form $\frac{C}{0}$. The constant $C = 1 > 0$ in this case, and the denominator changes sign from negative (left of 0) to positive (right of 0). Hence

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Example 5.5: We analyze the rational function

$$f(x) = \frac{x-2}{x^2-2x+1}.$$

Numerator: $x-2$ is < 0 when $x < 2$, $= 0$ when $x = 2$ and > 0 when $x > 2$.
Denominator: $x^2 - 2x + 1 = (x-1)^2$ is always ≥ 0, and $= 0$ only at $x = 1$.

At $x = 1$ we encounter the situation $\frac{C}{0}$ with $C < 0$. Near $x = 1$, the denominator is > 0, so that both left and right limits must be $-\infty$ and we have:

$$\lim_{x\to 1} \frac{x-2}{x^2-2x+1} = -\infty.$$

The rules we stated allow us to state the following important theorem:

Theorem 5.5 (Algebra of Limits)**:** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, and let x_0 either be a real number, or $+\infty$ or $-\infty$. Suppose that as $x \to x_0$, both f and g have limits

$$\lim_{x\to x_0} f(x) = \ell_f \qquad \text{and} \qquad \lim_{x\to x_0} f(x) = \ell_g$$

(these limits can be finite of infinite). Then the following equalities hold ***only*** *when the right hand side is a meaningful expression*:

$$\lim_{x\to x_0} (f(x) \pm g(x)) = \ell_f \pm \ell_g$$
$$\lim_{x\to x_0} (f(x) \cdot g(x)) = \ell_f \cdot \ell_g$$
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\ell_f}{\ell_g},$$

where in the last case, $g(x)$ must be nonzero for all x tending to x_0, and it is possible that there are different left and right limits (for instance, if g changes sign at x_0).

Proof. We prove just one case to demonstrate the idea of the proof. Let us prove that

$$\lim_{x\to x_0} (f(x) + g(x)) = \ell_f + \ell_g$$

in the case that both ℓ_f and ℓ_g are real numbers (and not $\pm\infty$). That is, we need to show that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $0 < |x - x_0| < \delta$, we have that $|(f(x) + g(x)) - (\ell_f + \ell_g)| < \varepsilon$.

Fix $\tilde{\varepsilon} > 0$. We start with f: since $\lim_{x\to x_0} f(x) = \ell_f$,

$$\exists \delta_f = \delta_f(\tilde{\varepsilon}) > 0 \text{ such that } \forall 0 < |x - x_0| < \delta_f,\ |f(x) - \ell_f| < \tilde{\varepsilon}.$$

Similarly, for g, using the same $\tilde{\varepsilon}$:

$$\exists \delta_g = \delta_g(\tilde{\varepsilon}) > 0 \text{ such that } \forall 0 < |x - x_0| < \delta_g,\ |g(x) - \ell_g| < \tilde{\varepsilon}.$$

Define $\delta = \min\{\delta_f, \delta_g\}$. Then for all $0 < |x - x_0| < \delta$,

$$\begin{aligned} |(f(x) + g(x)) - (\ell_f + \ell_g)| &= |(f(x) - \ell_f) + (g(x) - \ell_g)| \\ \textit{(triangle inequality)} \qquad &\le |f(x) - \ell_f| + |g(x) - \ell_g| \\ &< \tilde{\varepsilon} + \tilde{\varepsilon} = 2\tilde{\varepsilon}. \end{aligned}$$

By defining $\varepsilon = 2\tilde{\varepsilon}$ we are done: we have shown that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < |x - x_0| < \delta$, we have $\Big|\underbrace{(f(x) + g(x))}_{\text{this is the function}} - \underbrace{(\ell_f + \ell_g)}_{\text{this is the limit}}\Big| < \varepsilon$. □

An immediate corollary is this:

Corollary 5.6: If f and g are continuous at $x_0 \in \mathbb{R}$, then so are $f \pm g$, $f \cdot g$ and $\frac{f}{g}$ continuous at x_0 (the last one only if $g(x_0) \neq 0$).

Another corollary is this:

Corollary 5.7: Rational functions are continuous on their domains and polynomials are continuous on $\mathbb{R}$.

5.3 Comparison theorems

Theorem 5.8: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Suppose that

$$\lim_{x \to x_0} f(x) = \ell_f \qquad \text{and} \qquad \lim_{x \to x_0} g(x) = \ell_g.$$

If in a neighborhood of x_0 (excluding x_0 itself) we have $f(x) \le g(x)$, then $\ell_f \le \ell_g$.

Proof. We skip the proof here. It can be found in the book. □

Theorem 5.9 (Squeeze Theorem): Let $f, g, h : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell \in \mathbb{R} \qquad \text{and} \qquad f \le g \le h \text{ in a neighborhood of } x_0$$

(excluding x_0 itself)

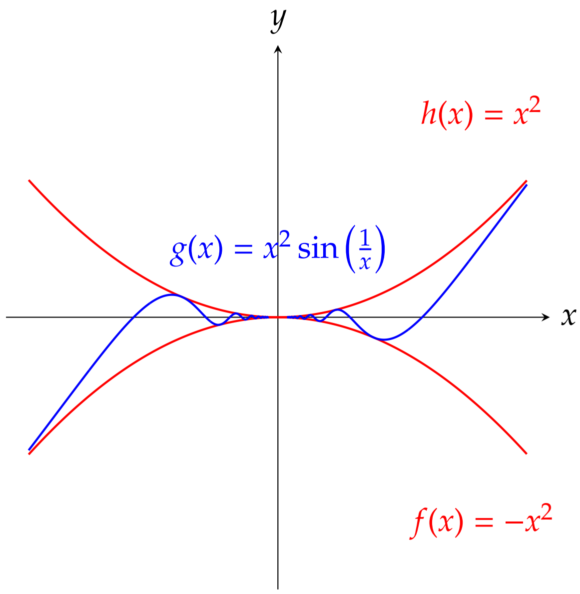
then

$$\lim_{x \to x_0} g(x) = \ell.$$

Proof. We skip the proof here. It can be found in the book. □

Example 5.6: Using this theorem we can immediately conclude that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$



Example 5.7: The Squeeze Theorem justifies our computation in Example 4.12 (the removable singularity of $\frac{\sin x}{x}$ at $x = 0$). In that problem we found that

$$\underbrace{\cos x}_{f(x)} < \underbrace{\frac{\sin x}{x}}_{g(x)} < \underbrace{1}_{h(x)}$$

and used the fact that $\lim_{x \to 0} \cos x = 1$ to conclude that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$