Finally, some really bad discontinuities!

We have seen removable discontinuities and jump discontinuities. Perhaps it would be wise to define what is a discontinuity is:

Discontinuity

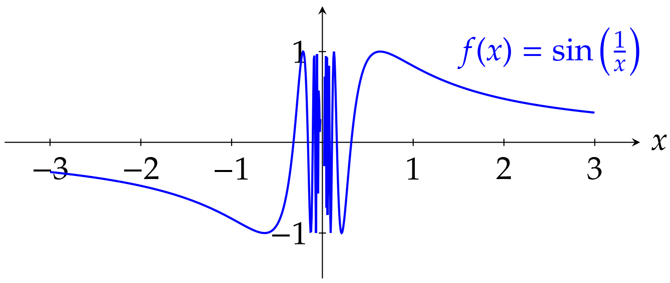
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If f is not continuous at x_0 then we say that it is **discontinuous at** x_0 and x_0 is called a **point of discontintinuity**.

There are points of discontinuity that are neither removable nor jump discontinuities:

Consider the function

$$f(x) = \sin\left(\frac{1}{x}\right)$$

whose domain is $\mathbb{R} \setminus \{0\}$. It is discontinuous at $x_0 = 0$ because the limit does not exist: indeed, as x approaches 0, the argument $\frac{1}{x}$ grows without bound, causing the sine function to oscillate infinitely rapidly between -1 and 1. No matter how small a δ-neighborhood around $x_0 = 0$ we choose, the function takes all values between -1 and 1 infinitely many times, preventing convergence to any particular limit value.



Discontinuity of the second type

A discontinuity point that is neither removable nor jump, is called a **discontinuity of the second type**.

Limits of monotone functions

The situation is better for monotone functions, just as it was for monotone sequences:

Theorem 4.5: A monotone (increasing or decreasing) function $f : \mathbb{R} \to \mathbb{R}$ cannot have a discontinuity of the second type. That is, a monotone function could only have removable discontinuities, jump discontinuities, or have asymptotes (vertical or horizontal).

Proof. We prove the theorem for a monotone increasing function. The same ideas will carry over for a monotone decreasing function. We split the proof into two claims:

(1) Claim: for any $x_0 \in \{-\infty\} \cup \mathbb{R}$,

$$\lim_{x \to x_0^+} f(x) = \inf_{x > x_0} f(x).$$

Let $L_+ = \inf_{x>x_0} f(x)$ and suppose that $L_+ \in \mathbb{R}$. By the definition of the infimum, for any $\varepsilon > 0$, there exists $x_1 > x_0$ such that $f(x_1) < L_+ + \varepsilon$. Since f is monotone increasing, for all $x \in (x_0, x_1)$, we have $L_+ \leq f(x) \leq f(x_1) < L_+ + \varepsilon$. Thus, $|f(x) - L_+| < \varepsilon$ whenever $0 < x - x_0 < x_1 - x_0$, proving the right-hand limit exists and equals L_+. A similar idea proves the claim for $L_+ = -\infty$.

(2) Claim: for any $x_0 \in \mathbb{R} \cup \{+\infty\}$,

$$\lim_{x\to x_0^-} f(x) = \sup_{x<x_0} f(x).$$

Let $L_- = \sup_{x<x_0} f(x)$. By the definition of the supremum, for any $\varepsilon > 0$, there exists $x_1 < x_0$ such that $f(x_1) > L_- - \varepsilon$. Since f is increasing, for all $x \in (x_1, x_0)$, we have $L_- - \varepsilon < f(x_1) \leq f(x) \leq M$. Thus, $|f(x) - L_-| < \varepsilon$ whenever $0 < x_0 - x < x_0 - x_1$, proving the left-hand limit exists and equals L_-. A similar idea proves the claim for $L_- = +\infty$.

Hence, at any point $x_0 \in \mathbb{R}$, both one-sided limits exist (though they may be infinite). The only possible discontinuities are:

- **Removable discontinuity**: when $L_- = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L_+$.
- **Jump discontinuity**: when $L_- = \lim_{x\to x_0^-} f(x) < \lim_{x\to x_0^+} f(x) = L_+$
- **Vertical asymptote**: when one of the one-sided limits is infinite (then the other one will not exist because of monotonicity): $L_+ = -\infty$ or $L_- = +\infty$.

A discontinuity of the second type cannot occur. □

Corollary 4.6: Let $f : \mathbb{R} \to \mathbb{R}$ be monotone increasing. Then for any $x_0 \in \mathbb{R}$, if f is defined in a neighborhood of x_0 (but not necessarily at x_0),

$$\lim_{x\to x_0^-} f(x) \leq \lim_{x\to x_0^+} f(x)$$

If f is defined at x_0, then

$$\lim_{x\to x_0^-} f(x) \leq f(x_0) \leq \lim_{x\to x_0^+} f(x).$$

An analogous statement holds for a monotone decreasing function.

Proof. This is an immediate consequence of Theorem 4.5. □

Chapter 5

Properties and computation of limits

5.1 Uniqueness of the limit and local sign of a function

Uniqueness

We always write *the* limit, not *a* limit. Implicitly, we say that it is *unique*. This is true, however it requires proof. Here is the formal statement (an analogous statement could be made for sequences):

Theorem 5.1 (Uniqueness of limits): Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Suppose that $\lim_{x \to x_0} f(x) = \ell$, where ℓ could be finite of infinite. Then there can be no limit other than ℓ as $x \to x_0$.

Proof. Exercise. *Hint: by contradiction.* □

Local sign

It is intuitively clear that if a function has a positive limit (or $+\infty$), then as we approach this limit the values of the function must also be positive. Analogously, if a limit is negative (or $-\infty$), then the values nearby should be negative. This is stated as follows:

Theorem 5.2 (Local sign): Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$.

- If $\lim_{x \to x_0} f(x) > 0$ or $\lim_{x \to x_0} f(x) = +\infty$
  then $f > 0$ on a neighborhood of x_0 (potentially excluding x_0 itself).
- If $\lim_{x \to +\infty} f(x) > 0$ or $\lim_{x \to +\infty} f(x) = +\infty$
  then there exists $M > 0$ s.t. $f > 0$ on $\{x > M\}$.
- If $\lim_{x \to -\infty} f(x) > 0$ or $\lim_{x \to -\infty} f(x) = +\infty$
  then there exists $M < 0$ s.t. $f > 0$ on $\{x < M\}$.

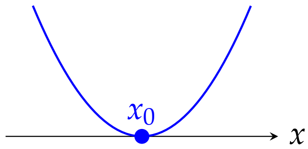
Analogous statements hold if these limits are negative.

Proof. We prove the first claim: $\lim_{x \to x_0} f(x) > 0 \quad \Rightarrow \quad f > 0$ on a neighborhood of x_0. Let $\ell = \lim_{x \to x_0} f(x) > 0$. Let $\varepsilon = \frac{\ell}{2} > 0$. By the definition of the limit, there exists $\delta = \delta(\varepsilon) > 0$ such that for $0 < |x - x_0| < \delta$

$$f(x) \in (\ell - \varepsilon, \ell + \varepsilon) = \left(\ell - \frac{\ell}{2}, \ell + \frac{\ell}{2}\right) = \left(\frac{\ell}{2}, \frac{3\ell}{2}\right) \subset (0, +\infty).$$

Hence $f > 0$ on this neighborhood of x_0 (potentially excluding x_0 itself), which completes the proof. The other claims in the theorem are proved in a similar way. □

The converse of this theorem is *almost* true. As the figure below shows, we can have situations where for all x satisfying $0 < |x - x_0| < \delta$ (for some $\delta > 0$ small), $f(x) > 0$, and yet $f(x_0) = 0$.



Hence we can prove the following statement, which is not quite the converse of the previous theorem:

Theorem 5.3: Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Assume that $\lim_{x\to x_0} f(x)$ exists.

- If $f \geq 0$ on a neighborhood of x_0
  then $\lim_{x\to x_0} f(x) \geq 0$ or $\lim_{x\to x_0} f(x) = +\infty$.
- If there exists $M > 0$ s.t. $f \geq 0$ on $\{x > M\}$
  then $\lim_{x\to+\infty} f(x) \geq 0$ or $\lim_{x\to+\infty} f(x) = +\infty$
- If there exists $M < 0$ s.t. $f \geq 0$ on $\{x < M\}$
  then $\lim_{x\to-\infty} f(x) \geq 0$ or $\lim_{x\to-\infty} f(x) = +\infty$.

Analogous statements hold if these limits are negative.

Proof. We prove the first claim (the others follow a similar strategy). By contradiction, assume that $f \geq 0$ on a neighborhood of x_0 and that $\lim_{x\to x_0} f(x) < 0$ or $\lim_{x\to x_0} f(x) = -\infty$. We immediately obtain a contradiction to Theorem 5.2. □

Theorem 5.4 (Local boundedness)**:** Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$.

- If $\lim_{x\to x_0} f(x)$ exists and is finite, then f is bounded on a neighborhood of x_0: there exist $\delta > 0$ and $A > 0$ such that for all $0 < |x - x_0| < \delta$, $|f(x)| < A$.
- If $\lim_{x\to+\infty} f(x)$ exists and is finite, then f is bounded for all large x: there exist $A > 0$ and $M > 0$ such that for all $x > M$, $|f(x)| < A$.
- If $\lim_{x\to-\infty} f(x)$ exists and is finite, then f is bounded for all large negative x: there exist $M < 0$ and $A > 0$ such that for all $x < M$, $|f(x)| < A$.

Proof. We prove the first claim. Denote $\ell = \lim_{x\to x_0} f(x) \in \mathbb{R}$. By definition of the limit, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $0 < |x - x_0| < \delta$, we have $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$. Choosing $A = |\ell| + \varepsilon$ will do the job: $|f(x)| < A$ for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. □