Let us prove the indeed this is the case for $f(x) = \frac{1}{(x-2)^2}$:

Example 4.11: Show that $f(x) = \frac{1}{(x-2)^2}$ tends to $+\infty$ as $x \to 2$.

We want to show that for every $A > 0$, there exists $\delta > 0$ such that:

$$0 < |x-2| < \delta \Rightarrow \frac{1}{(x-2)^2} > A.$$

Fix $A > 0$. We need to find $\delta > 0$ such that:

$$\frac{1}{(x-2)^2} > A$$

This inequality is equivalent to:

$$(x-2)^2 < \frac{1}{A}$$

Taking square roots (and noting both sides are positive):

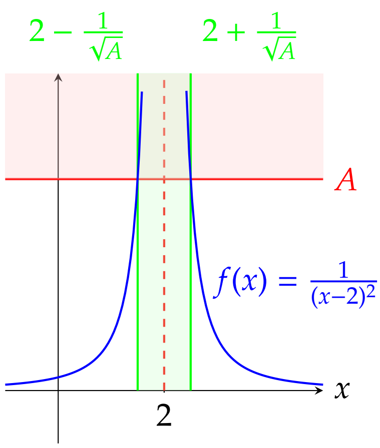
$$|x-2| < \frac{1}{\sqrt{A}}$$

Therefore, if we choose $\delta = \frac{1}{\sqrt{A}}$, then for $0 < |x-2| < \delta$:

$$\frac{1}{(x-2)^2} > \frac{1}{\delta^2} = A.$$

Since $A > 0$ was arbitrary, we conclude that

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty.$$



Negative infinite limit at x_0 (vertical asymptote)

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined in a neighborhood of some point $x_0 \in \mathbb{R}$, but possibly not at x_0 itself. We say that f **tends to** $-\infty$ **as** $x \to x_0$ if for every $A < 0$ there exists $\delta = \delta(A) > 0$ such that for all $0 < |x - x_0| < \delta$, we have $f(x) < A$, and we write

$$\lim_{x \to x_0} f(x) = -\infty.$$

The condition for convergence can be written symbolically as:

$$\forall A < 0,\ \exists \delta > 0,\ \text{s.t. } \forall 0 < |x - x_0| < \delta,\ f(x) < A.$$

In this case we also say that the line $x = x_0$ is a **vertical asymptote** of $f(x)$.

Note that as defined here, the infinite limit must be the same whether x tends to x_0 from the right or from the left. Hence, the function $f(x) = \frac{1}{x}$ does *not* have a limit as $x \to 0$.

Left and right limits and discontinuity points

We have already seen some examples of functions that have points where we tend to different values if we approach from the left or from the right. Two simple examples at the functions $f(x) = \frac{1}{x}$ and the ceiling function $g(x) = \lceil x \rceil$, sketched below.

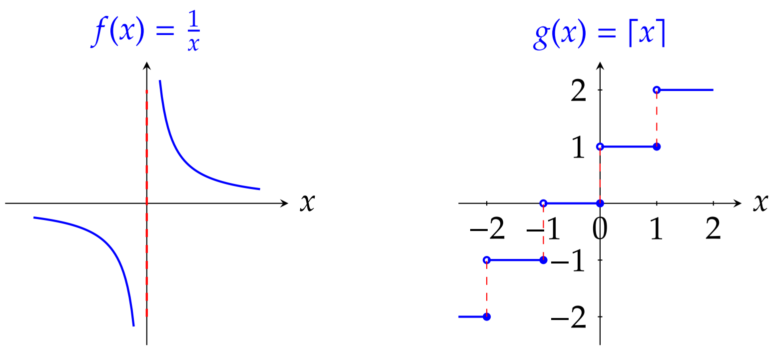


Figure 4.4: Examples of functions that have points with different left and right limits.

Hence we want to repeat the ideas that we've seen above, with the only difference being the neighborhoods around x_0: we'll want a *right* neighborhood and a *left* neighborhood.

Finite right limit

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f **has a right limit** ℓ **at** $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in (x_0, x_0 + \delta)$, we have that $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$ and we write

$$\lim_{x \to x_0^+} f(x) = \ell.$$

The condition for a right limit can be written symbolically as:

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \text{s.t. } 0 < x - x_0 < \delta \ \Rightarrow\ |f(x) - \ell| < \varepsilon.$$

Right-continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **right-continuous at** x_0 if

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

$\pm\infty$ right limit

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f **tends to** $\pm\infty$ **from the right at** $x_0 \in \mathbb{R}$ if for every $A \gtrless 0$ there exists $\delta = \delta(A) > 0$ such that for all $x \in (x_0, x_0 + \delta)$, we have that $f(x) \gtrless A$ and we write

$$\lim_{x \to x_0^+} f(x) = \pm\infty.$$

The condition for an infinite right limit can be written symbolically as:

$$\forall A \gtrless 0,\ \exists \delta = \delta(A) > 0,\ \text{s.t. } 0 < x - x_0 < \delta \ \Rightarrow\ f(x) \gtrless A.$$

Finite left limit

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f **has a left limit** ℓ **at** $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in (x_0 - \delta, x_0)$, we have that $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$ and we write

$$\lim_{x \to x_0^-} f(x) = \ell.$$

The condition for a left limit can be written symbolically as:

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \text{s.t. } 0 < x_0 - x < \delta \ \Rightarrow\ |f(x) - \ell| < \varepsilon.$$

Left-continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **left-continuous at** x_0 if

$$\lim_{x \to x_0^-} f(x) = f(x_0).$$

$\pm\infty$ left limit

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f **tends to** $\pm\infty$ **from the left at** $x_0 \in \mathbb{R}$ if for every $A \gtrless 0$ there exists $\delta = \delta(A) > 0$ such that for all $x \in (x_0 - \delta, x_0)$, we have that $f(x) \gtrless A$ and we write

$$\lim_{x \to x_0^-} f(x) = \pm\infty.$$

The condition for an infinite left limit can be written symbolically as:

$$\forall A \gtrless 0, \ \exists \delta = \delta(A) > 0, \text{ s.t. } 0 < x_0 - x < \delta \ \Rightarrow \ f(x) \gtrless A.$$

Looking back at Figure 4.4 it appears that the function $f(x) = \frac{1}{x}$ has limits $\pm\infty$ left/right limits, and that the function $g(x) = \lceil x \rceil$ has different left/right limits at all integer points, though it appears that it is always left-continuous.

Jump discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If at some point x_0 the left and right limits exist, yet

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$$

then we say that f has a **jump discontinuity** at x_0.

Conversely, we have the following simple proposition:

Proposition 4.4: Let $f : \mathbb{R} \to \mathbb{R}$ be defined in a neighborhood of x_0 (possibly not at x_0 itself). Then

$$\lim_{x \to x_0} f(x) = L \qquad \Leftrightarrow \qquad \lim_{x \to x_0^+} f(x) = L \quad \text{and} \quad \lim_{x \to x_0^-} f(x) = L$$

where L can be any number or $\pm\infty$. Moreover, the function is continuous at x_0 if and only if it is both right- and left-continuous at x_0.

Proof. Exercise. $\square$

Now we have the tools to consider the removable discontinuity of $f(x) = \frac{\sin x}{x}$ at $x = 0$:

Example 4.12: Consider the function $f(x) = \frac{\sin x}{x}$ for $x \neq 0$. Show that f has a removable discontinuity at $x = 0$ and find the limit.

We want to show that $\lim_{x\to 0} \frac{\sin x}{x}$ exists. We guess that as $x \to 0$, the value of $f(x)$ should tend to 1 (there are good reasons for this particular guess, which we will see later in the course). We want to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Consider the unit circle and let x be a small positive angle. From geometric considerations, for $0 < x < \frac{\pi}{2}$ we have:

$$\sin x < x < \tan x$$

(the first inequality is relatively simple, the second requires a bit more work). Dividing by $\sin x$ (which is positive, so inequalities don't change direction):

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals (which reverses inequalities):

$$\cos x < \frac{\sin x}{x} < 1.$$

Since $\cos x$ is continuous and $\cos 0 = 1$, the only possible option (we will prove this later, it is called *the squeezing theorem*) is that:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

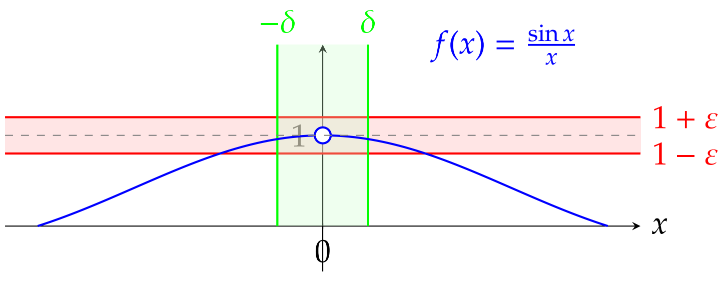
For $x < 0$, let $y = -x > 0$, then:

$$\frac{\sin x}{x} = \frac{\sin(-y)}{-y} = \frac{-\sin y}{-y} = \frac{\sin y}{y}$$

So the left limit equals the right limit:

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Therefore, using Proposition 4.4, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and the discontinuity at $x = 0$ is removable.



Finally, some really bad discontinuities!

We have seen removable discontinuities and jump discontinuities. Perhaps it would be wise to define what is a discontinuity is:

Discontinuity

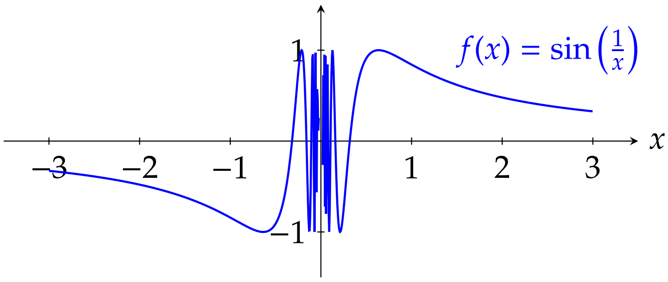
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If f is not continuous at x_0 then we say that it is **discontinuous at** x_0 and x_0 is called a **point of discontintinuity**.

There are points of discontinuity that are neither removable nor jump discontinuities:

Consider the function

$$f(x) = \sin\left(\frac{1}{x}\right)$$

whose domain is $\mathbb{R} \setminus \{0\}$. It is discontinuous at $x_0 = 0$ because the limit does not exist: indeed, as x approaches 0, the argument $\frac{1}{x}$ grows without bound, causing the sine function to oscillate infinitely rapidly between -1 and 1. No matter how small a δ-neighborhood around $x_0 = 0$ we choose, the function takes all values between -1 and 1 infinitely many times, preventing convergence to any particular limit value.



Discontinuity of the second type

A discontinuity point that is neither removable nor jump, is called a **discontinuity of the second type**.

Limits of monotone functions

The situation is better for monotone functions, just as it was for monotone sequences:

Theorem 4.5: A monotone (increasing or decreasing) function $f : \mathbb{R} \to \mathbb{R}$ cannot have a discontinuity of the second type. That is, a monotone function could only have removable discontinuities, jump discontinuities, or have asymptotes (vertical or horizontal).

Proof. We prove the theorem for a monotone increasing function. The same ideas will carry over for a monotone decreasing function. We split the proof into two claims:

(1) Claim: for any $x_0 \in \{-\infty\} \cup \mathbb{R}$,

$$\lim_{x \to x_0^+} f(x) = \inf_{x > x_0} f(x).$$