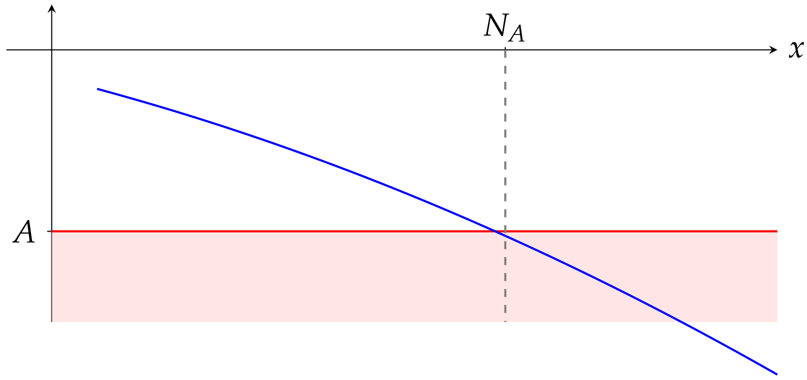


Limits at negative infinity ($x \to -\infty$)

The previous definitions can all be modified to consider limits of functions as x tends to $-\infty$. We omit these here, but show an example:

Example 4.7: The function $f(x) = \frac{3x+2}{x-1}$ tends to 3 as $x \to -\infty$.

Fix $\varepsilon > 0$. We need to find $N_\varepsilon \in \mathbb{R}$ such that for all $x < N_\varepsilon$, we have $\left|3 - \frac{3x+2}{x-1}\right| < \varepsilon$. First, simplify the expression:

$$\begin{aligned}\left|3 - \frac{3x+2}{x-1}\right| &= \left|\frac{3(x-1)-(3x+2)}{x-1}\right| \\ &= \left|\frac{3x-3-3x-2}{x-1}\right| \\ &= \left|\frac{-5}{x-1}\right| \\ &= \frac{5}{|x-1|}.\end{aligned}$$

For $x < 0$, we have $|x-1| = 1 - x > -x > 0$, so:

$$\frac{5}{|x-1|} < \frac{5}{-x} = -\frac{5}{x}.$$

We want $-\frac{5}{x} < \varepsilon$, which for $x < 0$ is equivalent to $-x > \frac{5}{\varepsilon}$, or $x < -\frac{5}{\varepsilon}$. Take $N_\varepsilon = -\frac{5}{\varepsilon}$. Then for any $x < N_\varepsilon$, we have:

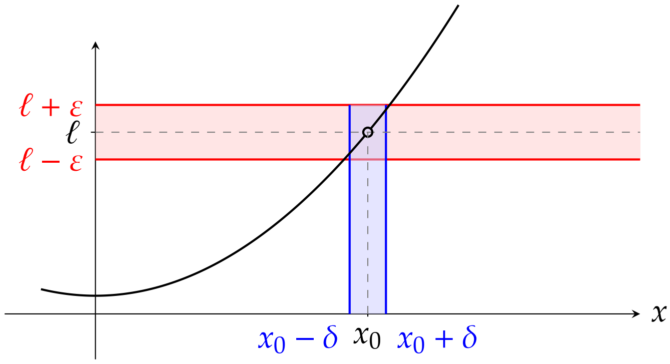
$$\left|3 - \frac{3x+2}{x-1}\right| < -\frac{5}{x} < -\frac{5}{N_\varepsilon} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that for every $\varepsilon > 0$, there exists N_ε such that for all $x < N_\varepsilon$, $\left|3 - f(x)\right| < \varepsilon$. Therefore,

$$\lim_{x \to -\infty} \frac{3x+2}{x-1} = 3.$$

Finite limits and continuity

When we want to study the properties of a function $f : \mathbb{R} \to \mathbb{R}$ at a given point $x_0 \in \mathbb{R}$, we want to understand how it behaves for other points $x \neq x_0$ that are close to x_0. Our method is similar to what we've seen before. We call $\ell \in \mathbb{R}$ the 'suspected' value of f at x_0, and consider an ε-neighborhood of ℓ. We then ask whether a small neighborhood of x_0 is within the pre-image of this neighborhood. We note that it isn't necessarily the case that $\ell = f(x_0)$ (we will see several such scenarios). However, if $\ell = f(x_0)$ then the function is continuous at x_0.



$\varepsilon - \delta$ formulation

The so-called $\varepsilon - \delta$ formulation is a staple of modern analysis: $\varepsilon > 0$ measures a small permissible margin of error along the y-axis, and, corresponding to it is $\delta > 0$ (*which depends upon* ε), measuring the corresponding allowable points of input along the x-axis.

Point of continuity

In the case that $\ell = f(x_0)$ and the function approaches $f(x_0)$ for points x near x_0, then we say that the function is continuous at x_0:

Point of continuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f **is continuous at** $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ (depending on ε) such that for all $x \in (x_0 - \delta, x_0 + \delta)$, we have that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

The condition for continuity can be written symbolically as:

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \text{s.t. } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

In this case we write

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Example 4.8: We prove that $f(x) = x^2$ is continuous at $x_0 = 2$.

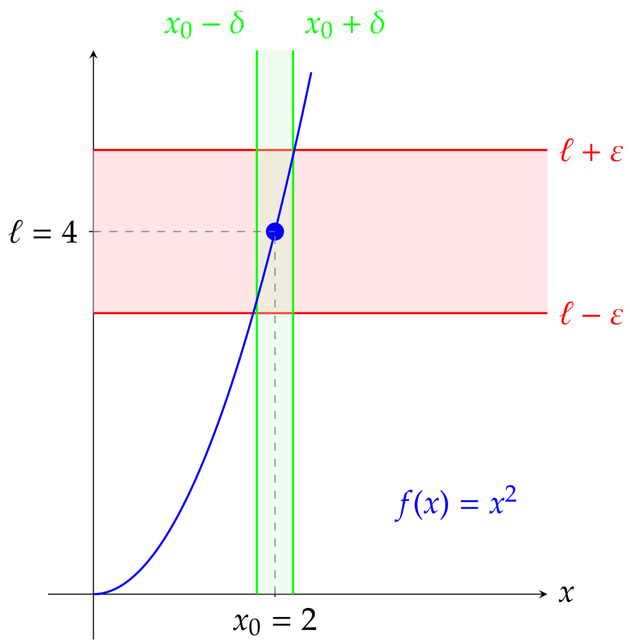
Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that:

$$|x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$$

Note that $|x^2-4| = |x-2||x+2|$. If we restrict $|x-2| < 1$, then $1 < x < 3$, so $|x+2| < 5$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$. Then if $|x-2| < \delta$:

$$|x^2-4| = |x-2||x+2| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon$$

Thus, $f(x) = x^2$ is continuous at $x_0 = 2$ with $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$.



Example 4.9: We prove that $f(x) = \cos x$ is continuous at every point $x_0 \in \mathbb{R}$.

Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that:

$$|x - x_0| < \delta \Rightarrow |\cos x - \cos x_0| < \varepsilon$$

Using the trigonometric identity:

$$\cos x - \cos x_0 = -2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)$$

Taking absolute values and using the fact that $|\sin\theta| \leq |\theta|$ (this is a simple geometric fact):

$$|\cos x - \cos x_0| = 2\left|\sin\left(\frac{x+x_0}{2}\right)\right|\left|\sin\left(\frac{x-x_0}{2}\right)\right| \leq 2 \cdot 1 \cdot \left|\frac{x-x_0}{2}\right| = |x - x_0|$$

Thus, if we choose $\delta = \varepsilon$, then:

$$|x - x_0| < \delta = \varepsilon \Rightarrow |\cos x - \cos x_0| \leq |x - x_0| < \varepsilon$$

Therefore, $f(x) = \cos x$ is continuous at x_0 with $\delta = \varepsilon$.

Continuity on a set

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, and let $I \subseteq \mathrm{dom}(f)$. If f is continuous at every point $x \in I$, then we say that f **is continuous on the set** I.

Proposition 4.3: All the elementary functions that we've seen in Section 2.6 are continuous on their entire domains.

Proof. We have just shown this for the cosine. The sine function is proved in similar fashion. For the other elementary functions (powers, polynomials, rational functions, the other trigonometric functions and their inverses, exponential and logarithms) we postpone the proof until a later time. □

Removable discontinuity

Another scenario, is either if $f(x_0) \neq \ell$ or if $f(x_0)$ is not defined (i.e. x_0 isn't in the domain of f). In this case we can only say that f has a limit as $x \to x_0$, but that limit does not equal $f(x_0)$ (either because $f(x_0)$ is a different value, or because it is not defined).

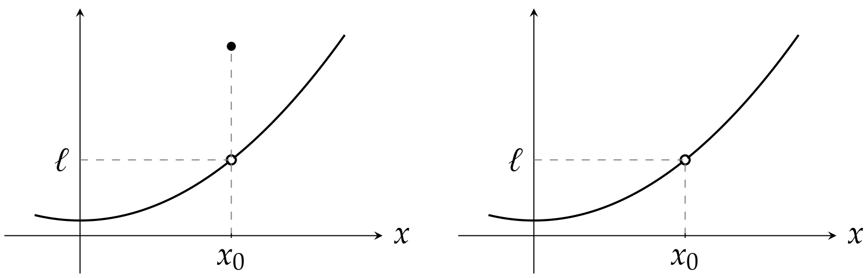


Figure 4.3: Left: $f(x_0) \neq \ell$, and right: f not defined at x_0.

To write the $\varepsilon - \delta$ definition we must exclude the point x_0 itself from consideration:

Removable discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that $f(x)$ **tends to** ℓ **as** $x \to x_0$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, we have that $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$.

The condition for convergence can be written symbolically as:

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ \text{s.t. } 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon.$$

In this case we write

$$\lim_{x \to x_0} f(x) = \ell.$$

If $f(x_0) = \ell$, then f is continuous at x_0, as we have previously defined. However, if $f(x_0) \neq \ell$ or if f is not defined at x_0, then we say that f **has a removable discontinuity at** x_0.

Example 4.10: We will show later that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$