- In the case that the sequence is monotone increasing:
  - ⋆ If $\sup\{a_n \mid n > N\} < +\infty$, then
  $$\lim_{n\to\infty} a_n = \sup\{a_n \mid n > N\}.$$
  - ⋆ If $\sup\{a_n \mid n > N\} = +\infty$, then the sequence diverges to $+\infty$.
- In the case that the sequence is monotone decreasing:
  - ⋆ If $\inf\{a_n \mid n > N\} > -\infty$, then
  $$\lim_{n\to\infty} a_n = \inf\{a_n \mid n > N\}.$$
  - ⋆ If $\inf\{a_n \mid n > N\} = -\infty$, then the sequence diverges to $-\infty$.

Proof. We only prove for the monotone *increasing* case (the decreasing case follows the same proof). For brevity we shall write

$$\sup_{n>N} a_n = \sup\{a_n \mid n > N\}$$

⋆ Suppose that $\sup_{n>N} a_n = \ell < +\infty$. Fix some $\varepsilon > 0$. By the definition of the supremum,

1. there exists some index $N_\varepsilon > N$ such that $\ell - a_{N_\varepsilon} < \varepsilon$;
2. for all $n > N$, $a_n \leq \ell$.

Combining these with the fact that the sequence is monotone increasing for $n > N$, we have the following sequence of inequalities

$$a_{N+1} \leq a_{N+2} \leq \cdots \leq \underbrace{a_{N_\varepsilon}}_{>\ell-\varepsilon} \leq a_{N_\varepsilon+1} \leq \cdots \leq \ell$$

Neglecting the terms up to a_{N_ε}, this can be written as

$$\ell - \varepsilon < a_{N_\varepsilon} \leq a_{N_\varepsilon+1} \leq \cdots \leq \ell$$

This means that for all $n \geq N_\varepsilon$, $|\ell - a_n| < \varepsilon$. By the definition of the limit of a sequence, this means that

$$\lim_{n\to\infty} a_n = \ell.$$

⋆ Now, suppose that $\sup_{n>N} a_n = +\infty$. Then (by definition) for every $A > 0$, there exists $N_A > N$ such that $a_{N_A} > A$. So we have

$$A < a_{N_A} \leq a_{N_A+1} \leq \cdots$$

By definition, this precisely means that

$$\lim_{n\to\infty} a_n = +\infty.$$

□

Example 4.3: 1. The sequence $a_n = \frac{n}{n+1}$ ($n \in \mathbb{N}$) is monotonically increasing, and its supremum is 1. Hence its limit exists (and it is 1)l.

2. The sequence $a_n = \frac{1}{n}$ ($n \in \mathbb{N}_+$) is monotonically decreasing and its infimum is 0. Hence its limit exists, and it is also 0.

Proposition 4.2 (The number e)**:** The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$, is monotonically increasing and bounded from above. Hence it has a limit, which is denoted e (this is the famous Euler's number, and this is how it is defined):

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. We need to show that the sequence a_n is bounded and monotonically increasing.

The sequence is monotonically increasing. Actually, it is strictly increasing. We write

$$\begin{aligned} a_n = \left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} 1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \end{aligned}$$

Similarly, we can express a_{n+1} as:

$$a_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} 1 \cdot \left(1 - \frac{1}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right).$$

Comparing a_n and a_{n+1} we see that:

$$\begin{array}{rl} a_n = & \displaystyle\sum_{k=0}^{n} \frac{1}{k!} 1 \cdot \left(1 - \frac{1}{n}\right) \cdots\cdots \left(1 - \frac{k-1}{n}\right) \\ & \qquad\qquad\quad \wedge \qquad\qquad\qquad \wedge \\ a_{n+1} = & \displaystyle\sum_{k=0}^{n+1} \frac{1}{k!} 1 \cdot \left(1 - \frac{1}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right) \end{array}$$

each term in the product is bigger in the expression for a_{n+1} (and, moreover, a_{n+1} has an additional positive summand $k = n+1$). Therefore $a_{n+1} > a_n$ (strict inequality).

The sequence is bounded. Observe that $a_1 = 2$, so that 2 is a lower bound (the sequence is increasing). We will now show that 3 is an upper bound. We shall use the inequality

$$k! = \underbrace{k(k-1)(k-2)\cdots 2}_{k-1 \text{ terms}} \cdot 1 \geq \underbrace{2 \cdot 2 \cdots 2}_{k-1 \text{ times}} = 2^{k-1}.$$

We write a_n as before:

$$\begin{aligned}
a_n &= \sum_{k=0}^{n} \frac{1}{k!} 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \dots \cdot \left(1 - \frac{k-1}{n}\right) \\
&< \sum_{k=0}^{n} \frac{1}{k!} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \\
&\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \\
&= 1 + \sum_{k=0}^{n-1} \frac{1}{2^k}.
\end{aligned}$$

We know the formula for the partial sum of a geometric series:

$$\sum_{k=0}^{n-1} \frac{1}{2^k} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) < 2.$$

So we find that

$$a_n < 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} < 1 + 2 = 3.$$

□

4.3 Limits of functions

Limits at infinity ($x \to +\infty$)

Our first few definitions are very similar to definitions we've already seen for sequences:

Finite limit at infinity (horizontal asymptote)

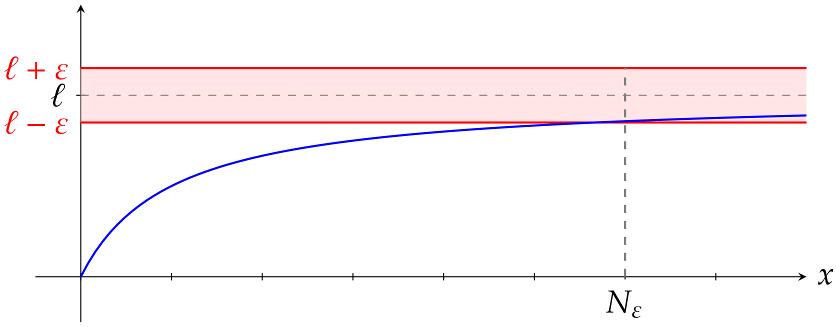
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If there exists $\ell \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{R}$ such that for all $x > N_\varepsilon$, $|\ell - f(x)| < \varepsilon$, we say that f tends to ℓ as $x \to +\infty$, and we write

$$\lim_{x \to +\infty} f(x) = \ell.$$

The condition for convergence can be written symbolically as:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{R},\ \text{s.t. } \forall x > N_\varepsilon,\ |\ell - f(x)| < \varepsilon.$$

In this case we say that the line $y = \ell$ is a **right horizontal asymptote** of $f(x)$.



Example 4.4: Let us show that the function $f(x) = \frac{1}{x}$ tends to 0 as $x \to +\infty$.

Fix $\varepsilon > 0$. We need to find $N_\varepsilon \in \mathbb{R}$ such that for all $x > N_\varepsilon$, we have $\left|0 - \frac{1}{x}\right| < \varepsilon$. Note that for $x > 0$, $\left|\frac{1}{x}\right| < \varepsilon$ is equivalent to $x > \frac{1}{\varepsilon}$. Take $N_\varepsilon = \frac{1}{\varepsilon}$. Then for any $x > N_\varepsilon$, we have:

$$\left|0 - \frac{1}{x}\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N_\varepsilon} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that for every $\varepsilon > 0$, there exists N_ε such that for all $x > N_\varepsilon$, $\left|0 - \frac{1}{x}\right| < \varepsilon$. Therefore,

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Example 4.5: Let us show that the function $f(x) = \frac{2x^2+3x-1}{x^2+1}$ tends to 2 as $x \to +\infty$.

Fix $\varepsilon > 0$. We need to find $N_\varepsilon \in \mathbb{R}$ such that for all $x > N_\varepsilon$, we have $\left|2 - \frac{2x^2+3x-1}{x^2+1}\right| < \varepsilon$.

First, simplify the expression:

$$\begin{aligned}\left|2 - \frac{2x^2 + 3x - 1}{x^2 + 1}\right| &= \left|\frac{2(x^2 + 1) - (2x^2 + 3x - 1)}{x^2 + 1}\right| \\ &= \left|\frac{2x^2 + 2 - 2x^2 - 3x + 1}{x^2 + 1}\right| \\ &= \left|\frac{-3x + 3}{x^2 + 1}\right| \\ &= \frac{3|x - 1|}{x^2 + 1}.\end{aligned}$$

For $x > 1$, we have $|x - 1| = x - 1 < x$, so:

$$\frac{3|x - 1|}{x^2 + 1} < \frac{3x}{x^2 + 1} < \frac{3x}{x^2} = \frac{3}{x}.$$

We want $\frac{3}{x} < \varepsilon$, which is equivalent to $x > \frac{3}{\varepsilon}$.

Take $N_\varepsilon = \max\left\{1, \frac{3}{\varepsilon}\right\}$ *(The* max *here is to ensure that N_ε is at least* 1*, which is a requirement from before)*. Then for any $x > N_\varepsilon$, we have:

$$\left|2 - \frac{2x^2 + 3x - 1}{x^2 + 1}\right| < \frac{3}{x} < \frac{3}{N_\varepsilon} \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that for every $\varepsilon > 0$, there exists N_ε such that for all $x > N_\varepsilon$, $|2 - f(x)| < \varepsilon$. Therefore,

$$\lim_{x \to +\infty} \frac{2x^2 + 3x - 1}{x^2 + 1} = 2.$$

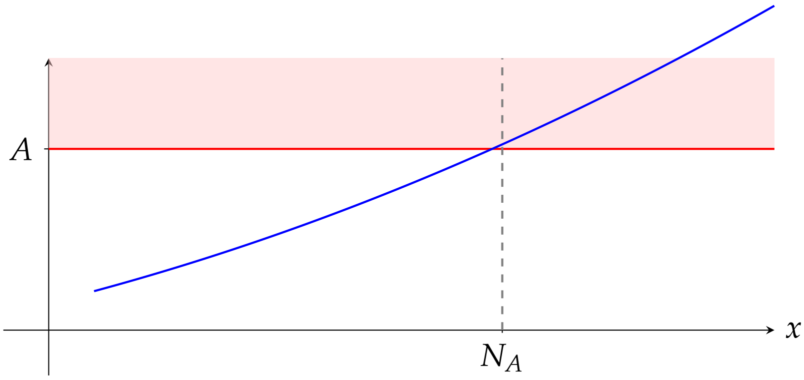
Positive infinite limit at infinity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If for any $A > 0$ there exists $N_A \in \mathbb{R}$ such that for all $x > N_A$, $f(x) > A$, we say that f tends to $+\infty$ as $x \to +\infty$, and we write

$$\lim_{x \to +\infty} f(x) = +\infty.$$

The condition for convergence can be written symbolically as:

$$\forall A > 0,\ \exists N_A \in \mathbb{R},\ \text{s.t.}\ \forall x > N_A,\ f(x) > A.$$



Example 4.6: We show that $f(x) = \ln(x)$ tends to $+\infty$ as $x \to +\infty$.

Fix $A > 0$. We need to find $N_A \in \mathbb{R}$ such that for all $x > N_A$, we have $\ln(x) > A$. Take $N_A = e^A$. Then for any $x > e^A$, we have:

$$\ln(x) > \ln(e^A) = A.$$

Since $A > 0$ was arbitrary, this shows that for every $A > 0$, there exists N_A such that for all $x > N_A$, $\ln(x) > A$. Therefore,

$$\lim_{x \to +\infty} \ln(x) = +\infty.$$

Negative infinite limit at infinity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. If for any $A < 0$ there exists $N_A \in \mathbb{R}$ such that for all $x > N_A$, $f(x) < A$, we say that f tends to $-\infty$ as $x \to +\infty$, and we write

$$\lim_{x \to +\infty} f(x) = -\infty.$$

The condition for convergence can be written symbolically as:

$$\forall A < 0,\ \exists N_A \in \mathbb{R},\ \text{s.t.}\ \forall x > N_A,\ f(x) < A.$$