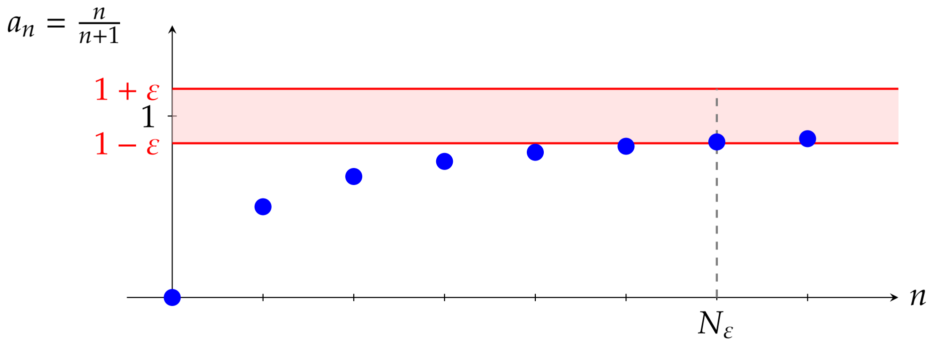


So here's the process:

- Fix $\varepsilon > 0$ ***(think of ε as being positive but VERY small!)***
- Compute $1 - a_n$:
$$1 - a_n = 1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}.$$
- Apply the condition $|1 - a_n| < \varepsilon$:
$$\frac{1}{n+1} < \varepsilon \quad \Leftrightarrow \quad 1 < \varepsilon(n+1) \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < n+1$$
- Find N_ε: define
$$N_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil.$$
- Verify the condition for $\forall n > N_\varepsilon$:
$$\forall n > N_\varepsilon, \qquad n+1 > N_\varepsilon + 1 = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 \geq \frac{1}{\varepsilon} + 1 > \frac{1}{\varepsilon}$$
$$\Leftrightarrow \qquad \frac{1}{n+1} < \varepsilon$$

We have therefore answered the previous question affirmatively:

A: *Yes,*
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \text{s.t. } \forall n > N_\varepsilon,\ |1 - a_n| < \varepsilon.$$
In fact, there is an explicit choice for N_ε, namely: $N_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil$.

Since this is true *for every* $\varepsilon > 0$, the sequence $a_n = \frac{n}{n+1}$ gets arbitrarily close to the number 1. We say that 1 is the *limit* of this sequence.

Finite limit of a sequence

Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. We say that the sequence **converges to** $\ell \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n > N_\varepsilon$, we have $|\ell - a_n| < \varepsilon$. In this case we write
$$\lim_{n\to\infty} a_n = \ell.$$
The condition for convergence can be written symbolically as:
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \text{s.t. } \forall n > N_\varepsilon,\ |\ell - a_n| < \varepsilon.$$

Returning to Example 4.1, we conclude that:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

A sequence might also grow indefinitely and 'tend' to $+\infty$ (for example, $a_n = e^n$), or decrease and 'tend' to $-\infty$ (for example, $a_n = -n^8$). This can be formalized as follows:

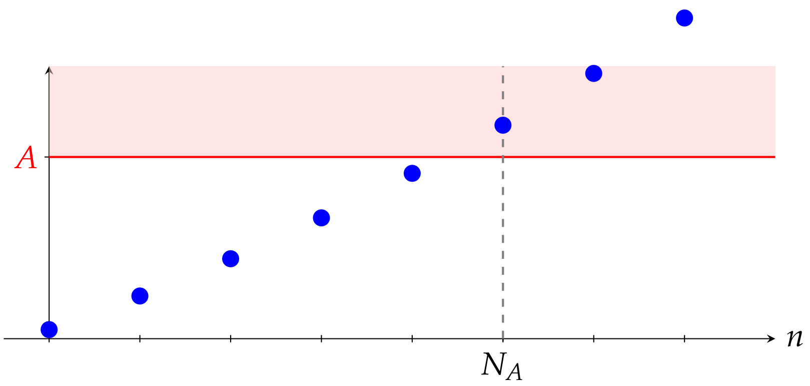
Divergent sequence (to $+\infty$)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. We say that the sequence **diverges to** $+\infty$ if for any $A > 0$ there exists $N_A \in \mathbb{N}$ such that for any $n > N_A$, we have $a_n > A$. In this case we write

$$\lim_{n\to\infty} a_n = +\infty.$$

The condition for convergence can be written symbolically as:

$$\forall A > 0,\ \exists N_A \in \mathbb{N},\ \text{s.t. } \forall n > N_A,\ a_n > A.$$



Here we ***think of** A **as being VERY big!***

We can make a similar definition for a sequence divergent to $-\infty$:

Divergent sequence (to $-\infty$)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. We say that the sequence **diverges to** $-\infty$ if for any $A < 0$ there exists $N_A \in \mathbb{N}$ such that for any $n > N_A$, we have $a_n < A$. In this case we write

$$\lim_{n\to\infty} a_n = -\infty.$$

The condition for convergence can be written symbolically as:

$$\forall A < 0,\ \exists N_A \in \mathbb{N},\ \text{s.t. } \forall n > N_A,\ a_n < A.$$

Here we ***think of** A **as being VERY big and negative!***

Example 4.2: Consider the sequence $a_n = \frac{n!}{n^{100}}$. Does $\{a_n\}_{n\in\mathbb{N}}$ have a limit as $n \to \infty$? If so, what is it? Does it diverge?

Solution. Let us rewrite the nth term as follows:

$$\begin{aligned} a_n &= \frac{n!}{n^{100}} \\ &= \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-98)\cdot(n-99)\cdot(n-100)\cdots3\cdot2\cdot1}{\underbrace{n\cdot n\cdots n}_{100\text{ times}}} \\ &= \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-98)\cdot(n-99)}{\underbrace{n\cdot n\cdots n}_{100\text{ times}}}\cdot(n-100)\cdots3\cdot2\cdot1. \end{aligned}$$

Observe that

$$\forall n > 198, \qquad \frac{n-99}{n} > \frac{1}{2}.$$

Hence, for all $n > 198$:

$$\begin{aligned} a_n &= \frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-98}{n}\cdot\frac{n-99}{n}\cdot(n-100)! \\ &> \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}\cdot\frac{1}{2}}_{100\text{ times}}\cdot(n-100)! \\ &= \frac{1}{2^{100}}\cdot(n-100)! \end{aligned}$$

Now, $\frac{1}{2^{100}}$ is some (*very* small) positive number, however $(n-100)!$ diverges as $n \to \infty$, so that eventually, for any $A > 0$, there exists $N_A \in \mathbb{N}$ such that $\frac{1}{2^{100}}\cdot(n-100)! > A$ for all $n > N_A$. Hence

$$\lim_{n\to\infty} a_n = +\infty.$$

Indeterminate sequences

Some sequences do not converge but also do not diverge. These are called **indeterminate sequences**. Examples include:

- The sequence $a_n = (-1)^n$ is *bounded* but does *not* converge.
- The sequence $a_n = n(-1)^n$ is *not* bounded (not from below and not from above) *and* does *not* converge.

A crucial feature of indeterminate sequences is that their values are not monotone: they increase and decrease repeatedly. The following theorem shows that this is indeed a crucial feature.

Theorem 4.1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. Assume that there exists $N \in \mathbb{N}$ such that for all $n > N$, the sequence is monotone. Then the sequence *cannot* be indeterminate. More precisely:

- In the case that the sequence is monotone increasing:
  - ⋆ If $\sup\{a_n \mid n > N\} < +\infty$, then
  $$\lim_{n\to\infty} a_n = \sup\{a_n \mid n > N\}.$$
  - ⋆ If $\sup\{a_n \mid n > N\} = +\infty$, then the sequence diverges to $+\infty$.
- In the case that the sequence is monotone decreasing:
  - ⋆ If $\inf\{a_n \mid n > N\} > -\infty$, then
  $$\lim_{n\to\infty} a_n = \inf\{a_n \mid n > N\}.$$
  - ⋆ If $\inf\{a_n \mid n > N\} = -\infty$, then the sequence diverges to $-\infty$.

Proof. We only prove for the monotone *increasing* case (the decreasing case follows the same proof). For brevity we shall write

$$\sup_{n>N} a_n = \sup\{a_n \mid n > N\}$$

⋆ Suppose that $\sup_{n>N} a_n = \ell < +\infty$. Fix some $\varepsilon > 0$. By the definition of the supremum,

1. there exists some index $N_\varepsilon > N$ such that $\ell - a_{N_\varepsilon} < \varepsilon$;
2. for all $n > N$, $a_n \leq \ell$.

Combining these with the fact that the sequence is monotone increasing for $n > N$, we have the following sequence of inequalities

$$a_{N+1} \leq a_{N+2} \leq \cdots \leq \underbrace{a_{N_\varepsilon}}_{>\ell-\varepsilon} \leq a_{N_\varepsilon+1} \leq \cdots \leq \ell$$

Neglecting the terms up to a_{N_ε}, this can be written as

$$\ell - \varepsilon < a_{N_\varepsilon} \leq a_{N_\varepsilon+1} \leq \cdots \leq \ell$$

This means that for all $n \geq N_\varepsilon$, $|\ell - a_n| < \varepsilon$. By the definition of the limit of a sequence, this means that

$$\lim_{n\to\infty} a_n = \ell.$$

⋆ Now, suppose that $\sup_{n>N} a_n = +\infty$. Then (by definition) for every $A > 0$, there exists $N_A > N$ such that $a_{N_A} > A$. So we have

$$A < a_{N_A} \leq a_{N_A+1} \leq \cdots$$

By definition, this precisely means that

$$\lim_{n\to\infty} a_n = +\infty.$$

□