3.4 Powers and nth roots

De Moivre's Formula

For any $n \in \mathbb{Z}$ and complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$:

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

This allows us to calculate the nth root of a complex number (now we restrict to $n \in \mathbb{N}$). Given $w = \rho e^{i\phi}$, we seek z such that $z^n = w$. Writing $z = re^{i\theta}$, we have:

$$r^n e^{in\theta} = \rho e^{i\phi} \quad \Rightarrow \quad r = \sqrt[n]{\rho} \quad \text{and} \quad n\theta = \phi + 2\pi k,\ k \in \mathbb{Z}$$

Since the complex exponential is 2π-periodic, we obtain n distinct roots with different arguments indexed as:

$$\theta_k = \frac{\phi + 2\pi k}{n}, \qquad k = 0, 1, \ldots, n-1.$$

nth roots

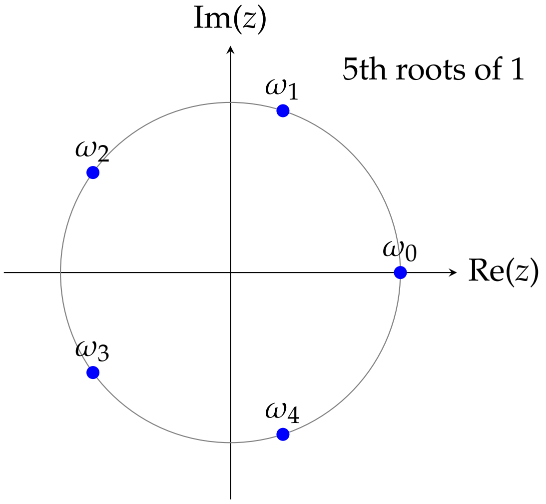
Given $w = \rho e^{i\phi}$, the n distinct nth roots are:

$$\sqrt[n]{w} = \sqrt[n]{\rho}\left(\cos\left(\frac{\phi + 2\pi k}{n}\right) + i\sin\left(\frac{\phi + 2\pi k}{n}\right)\right), \quad k = 0, 1, \ldots, n-1$$

In exponential form:

$$\sqrt[n]{w} = \sqrt[n]{\rho}\, e^{i\frac{\phi + 2\pi k}{n}}, \quad k = 0, 1, \ldots, n-1$$

The roots are equally spaced on a circle of radius $\sqrt[n]{\rho}$ in the complex plane.



Example 3.7: The cube roots of 1 are:

$$\omega_0 = \cos 0 + i\sin 0 = 1$$
$$\omega_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\omega_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Example 3.8: The fifth roots of 1 are:

$$\omega_0 = \cos 0 + i\sin 0 = 1$$
$$\omega_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$
$$\omega_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$
$$\omega_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$
$$\omega_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

Example 3.9: Find all cube roots of $z = 8i$. First write z in exponential form:

$$|z| = 8, \quad \arg(z) = \frac{\pi}{2}$$
$$z = 8e^{i\pi/2}$$

The cube roots are:

$$\omega_0 = \sqrt[3]{8}e^{i(\pi/2)/3} = 2e^{i\pi/6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3} + i$$
$$\omega_1 = 2e^{i(\pi/2+2\pi)/3} = 2e^{i5\pi/6} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -\sqrt{3} + i$$
$$\omega_2 = 2e^{i(\pi/2+4\pi)/3} = 2e^{i3\pi/2} = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -2i$$

Example 3.10: Compute $(1+i)^6$ using De Moivre's theorem:

$$|1+i| = \sqrt{2}, \quad \arg(1+i) = \frac{\pi}{4}$$
$$(1+i)^6 = (\sqrt{2})^6(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}) = 8(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 8(0-i) = -8i$$

Example 3.11: Solve $z^4 + 16 = 0$:

$$z^4 = -16 = 16e^{i\pi}$$
$$z = \sqrt[4]{16}e^{i(\pi+2\pi k)/4} = 2e^{i(\pi+2\pi k)/4}, \quad k = 0,1,2,3$$
$$z_0 = 2e^{i\pi/4} = \sqrt{2} + i\sqrt{2}$$
$$z_1 = 2e^{i3\pi/4} = -\sqrt{2} + i\sqrt{2}$$
$$z_2 = 2e^{i5\pi/4} = -\sqrt{2} - i\sqrt{2}$$
$$z_3 = 2e^{i7\pi/4} = \sqrt{2} - i\sqrt{2}$$

3.5 Algebraic equations and the Fundamental Theorem of Algebra

The study of polynomial equations leads to one of the most important results in mathematics: the Fundamental Theorem of Algebra. This theorem explains why complex numbers are not just a mathematical curiosity, but a necessary extension of the real number system that provides complete solutions to polynomial equations.

Fundamental Theorem of Algebra

Every non-constant polynomial of degree n with complex coefficients has exactly n complex roots (counting multiplicity).

Solving quadratic equations

For a quadratic equation $az^2 + bz + c = 0$ with real coefficients:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant $\Delta = b^2 - 4ac < 0$, the roots are complex conjugates.

Example 3.12: Solve $z^2 + 4z + 13 = 0$:

$$\begin{aligned} z &= \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} \\ &= \frac{-4 \pm 6i}{2} = -2 \pm 3i \end{aligned}$$

The roots are complex conjugates: $-2 + 3i$ and $-2 - 3i$.

Complex conjugate root theorem

If a polynomial with real coefficients has a complex root $a + bi$ ($b \neq 0$), then its complex conjugate $a - bi$ is also a root.

Example 3.13: The polynomial $z^3 - z^2 + z - 1 = 0$ has roots:

$$z = 1, \quad z = i, \quad z = -i.$$

Note that the complex roots i and $-i$ are conjugates.

Summary of Key Formulas

Property	Formula
Modulus	$\lvert z\rvert = \sqrt{a^2 + b^2}$
Argument	$\arg(z) = \arctan(b/a)$
Conjugate	$\bar{z} = a - bi$
Real part	$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$
Imaginary part	$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$
Polar form	$z = r(\cos\theta + i\sin\theta)$
Exponential form	$z = re^{i\theta}$
Multiplication	$z \cdot w = rs\, e^{i(\theta+\phi)}$
Division	$\frac{z}{w} = \frac{r}{s}\, e^{i(\theta-\phi)}$
De Moivre's	$z^n = r^n e^{in\theta}$
nth roots	$\sqrt[n]{z} = \sqrt[n]{r}\, e^{i(\theta+2\pi k)/n}$

Chapter 4

Limits and continuity

The concepts of limits and continuity lie at the very foundation of mathematical analysis. They formalize the intuitive ideas of *approaching* and *connectedness*, and provide the rigorous language needed to define derivatives and integrals.

Historically, these ideas evolved over centuries. Ancient Greek mathematicians like Eudoxus and Archimedes used rudimentary forms of limits in the *method of exhaustion* to compute areas and volumes. However, it was not until the 17th century, with the development of calculus by Newton and Leibniz, that the notion of a limit became central—though it remained informal and often controversial.

In the 19th century, mathematicians such as Cauchy and Weierstrass finally placed limits on a firm logical foundation using the famous ε-δ definitions. The key insight was to quantify the idea of "arbitrarily close" using two positive numbers: ε for the error tolerance in the function's output, and δ for the corresponding distance in the input. This rigorous framework eliminated the paradoxes and ambiguities that had plagued early calculus and remains the standard today.

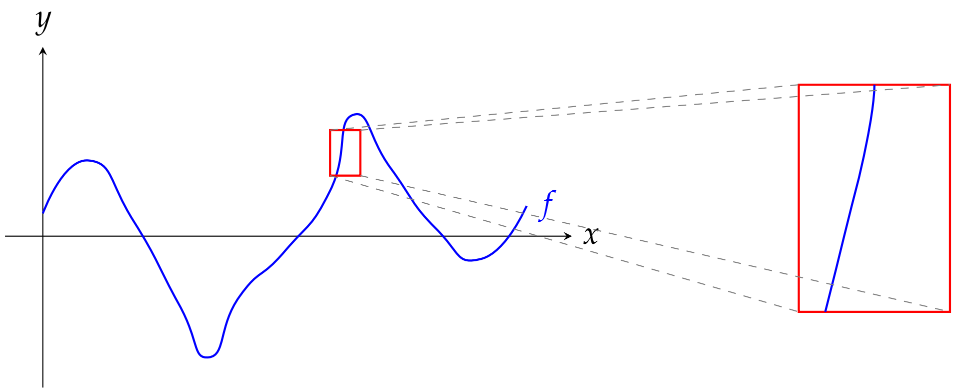


Figure 4.1: We will develop rigorous tools for "zooming-in"

4.1 Neighborhoods

To discuss continuity and limits, we need to be able to "zoom" into the graph of a function. To do this, we need a definition of a small subset around a given point that we want to investigate. This is called a *neighborhood*:

Neighborhood

Let $x_0 \in \mathbb{R}$ be a point on the real line and let $r > 0$ be a positive parameter. The **neighborhood of x_0 of radius r** is the *open* and bounded interval

$$I_r(x_0) = (x_0 - r, x_0 + r).$$

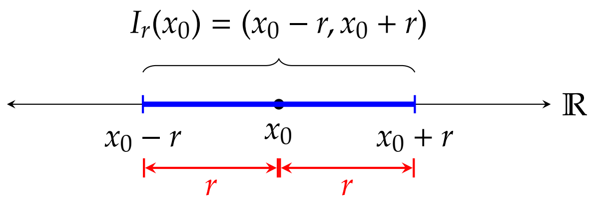


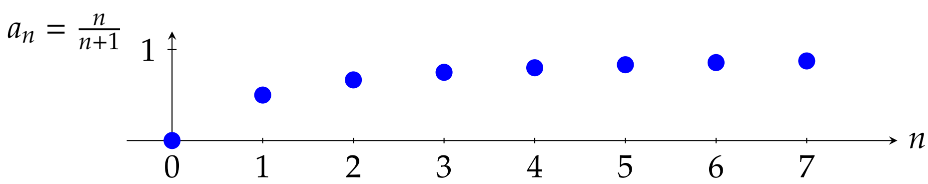
Figure 4.2: Neighborhood of x_0 with radius r

The idea will be that to understand what is happening at x_0, we shall take smaller and smaller neighborhoods of it, so that we can zoom in more and more.

4.2 Limits of sequences

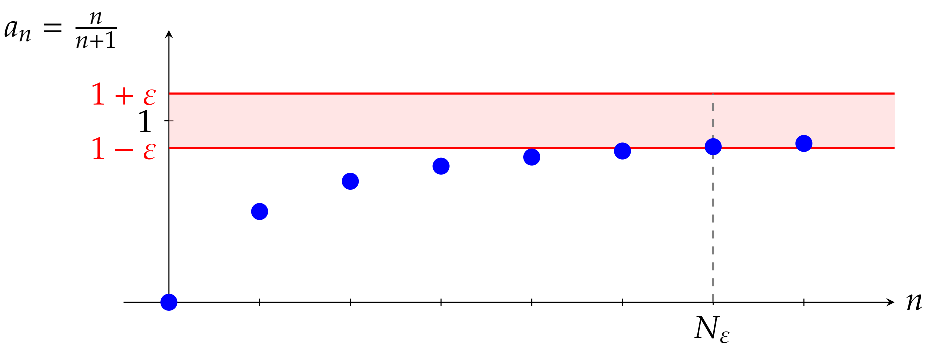
We start by considering sequences, which are functions $f : \mathbb{N} \to \mathbb{R}$. Consider the following two examples which we've seen before:

Example 4.1: Consider the sequence $a_n = \frac{n}{n+1}$.



It is evident that this sequence 'approaches' the number 1. However, we want to make this rigorous. To do this, we look at a small neighborhood of 1 and ask whether the sequence eventually enters this neighborhood (and never leaves!). So our question is this:

> **Q:** *For any $\varepsilon > 0$, does there exist $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$, $a_n \in (1 - \varepsilon, 1 + \varepsilon)$? Or, equivalently,*
>
> $$|1 - a_n| < \varepsilon, \qquad \forall n > N_\varepsilon?$$



So here's the process:

- Fix $\varepsilon > 0$ ***(think of ε as being positive but VERY small!)***
- Compute $1 - a_n$:
$$1 - a_n = 1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}.$$
- Apply the condition $|1 - a_n| < \varepsilon$:
$$\frac{1}{n+1} < \varepsilon \quad \Leftrightarrow \quad 1 < \varepsilon(n+1) \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < n+1$$
- Find N_ε: define
$$N_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil.$$
- Verify the condition for $\forall n > N_\varepsilon$:
$$\forall n > N_\varepsilon, \qquad n+1 > N_\varepsilon + 1 = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 \geq \frac{1}{\varepsilon} + 1 > \frac{1}{\varepsilon}$$
$$\Leftrightarrow \qquad \frac{1}{n+1} < \varepsilon$$

We have therefore answered the previous question affirmatively:

A: *Yes,*
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \text{s.t. } \forall n > N_\varepsilon,\ |1 - a_n| < \varepsilon.$$
In fact, there is an explicit choice for N_ε, namely: $N_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil$.

Since this is true *for every* $\varepsilon > 0$, the sequence $a_n = \frac{n}{n+1}$ gets arbitrarily close to the number 1. We say that 1 is the *limit* of this sequence.

Finite limit of a sequence

Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. We say that the sequence **converges to** $\ell \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n > N_\varepsilon$, we have $|\ell - a_n| < \varepsilon$. In this case we write
$$\lim_{n\to\infty} a_n = \ell.$$
The condition for convergence can be written symbolically as:
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \text{s.t. } \forall n > N_\varepsilon,\ |\ell - a_n| < \varepsilon.$$