Chapter 3

Complex Numbers

Complex numbers extend the real number system to solve equations that have no real solutions, such as $x^2 + 1 = 0$. They are essential in electrical engineering (AC circuit analysis, signal processing), mechanical engineering (vibrations, control systems), and many other fields. The most important mathematical property of complex numbers is that they provide a complete system where every non-constant polynomial equation has a solution. This is known as the **Fundamental Theorem of Algebra**. For example, the equation $x^2 + 1 = 0$ has no real solution, but has two complex solutions: $x = i$ and $x = -i$.

Definition

A **complex number** is an expression of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and i is the **imaginary unit** satisfying $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$.

- a is called the **real part** of z, denoted $\mathrm{Re}(z)$
- b is called the **imaginary part** of z, denoted $\mathrm{Im}(z)$
- If $b = 0$, then z is a real number
- If $a = 0$, then z is called a **purely imaginary** number

3.1 Algebraic operations

Algebraic operations

Let $z = a + bi$ and $w = c + di$ be complex numbers.

$$\begin{aligned}
\textbf{Addition:}\quad & z + w = (a + c) + (b + d)i \\
\textbf{Subtraction:}\quad & z - w = (a - c) + (b - d)i \\
\textbf{Multiplication:}\quad & z \cdot w = (ac - bd) + (ad + bc)i \\
\textbf{Division:}\quad & \frac{z}{w} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}, \quad w \neq 0 \\
\textbf{Inverse:}\quad & z^{-1} = \frac{1}{z} = \frac{a - bi}{a^2 + b^2}, \quad z \neq 0
\end{aligned}$$

Example 3.1: Let $z = 2 + 3i$ and $w = 1 - 2i$. Then:

$$\begin{aligned}
z + w &= (2 + 1) + (3 - 2)i = 3 + i \\
z \cdot w &= (2 \cdot 1 - 3 \cdot (-2)) + (2 \cdot (-2) + 3 \cdot 1)i = (2 + 6) + (-4 + 3)i = 8 - i \\
\frac{z}{w} &= \frac{(2 \cdot 1 + 3 \cdot (-2)) + (3 \cdot 1 - 2 \cdot (-2))i}{1^2 + (-2)^2} = \frac{(2 - 6) + (3 + 4)i}{5} = \frac{-4 + 7i}{5} \\
z^{-1} &= \frac{1}{2 + 3i} = \frac{2 - 3i}{2^2 + 3^2} = \frac{2 - 3i}{13} = \frac{2}{13} - \frac{3}{13}i
\end{aligned}$$

Example 3.2: Let $z = 4 - i$ and $w = 2 + 3i$. Compute z^2 and $\frac{w}{z}$:

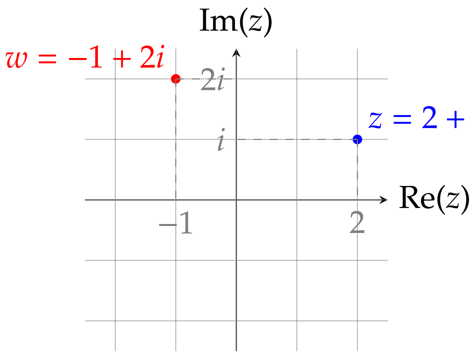
$$\begin{aligned}
z^2 &= (4 - i)^2 = 16 - 8i + i^2 = 16 - 8i - 1 = 15 - 8i \\
\frac{w}{z} &= \frac{2 + 3i}{4 - i} = \frac{(2 + 3i)(4 + i)}{(4 - i)(4 + i)} = \frac{(8 - 3) + (2 + 12)i}{4^2 - i^2} = \frac{5 + 14i}{17} = \frac{5}{17} + \frac{14}{17}i
\end{aligned}$$

3.2 Cartesian coordinates

Complex plane

The complex number $z = a + bi$ can be represented as a point (a, b) in the **complex plane**:

- The horizontal axis represents the real part ($\mathrm{Re}(z)$)
- The vertical axis represents the imaginary part ($\mathrm{Im}(z)$)



Key Properties of Complex Numbers

For $z = a + bi$:

- The **modulus of** z (absolute value) is

$$|z| = \sqrt{a^2 + b^2}.$$

- The **argument of** z is

$$\arg(z) = \theta$$

  where $\theta = \arctan \frac{b}{a}$.

- The **complex conjugate of** z is

$$\bar{z} = a - bi.$$

- The real and imaginary parts can be expressed as:

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

Example 3.3: For $z = 3 + 4i$:

$$\begin{aligned} |z| &= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \\ \arg(z) &= \arctan\left(\frac{4}{3}\right) \approx 0.9273 \text{ radians} \\ \bar{z} &= 3 - 4i \end{aligned}$$

We can also verify the real and imaginary parts (though we already know they are 3 and 4, respectively:

$$\begin{aligned} \mathrm{Re}(z) &= \frac{(3 + 4i) + (3 - 4i)}{2} = \frac{6}{2} = 3 \\ \mathrm{Im}(z) &= \frac{(3 + 4i) - (3 - 4i)}{2i} = \frac{8i}{2i} = 4 \end{aligned}$$

3.3 Trigonometric and exponential form

Polar form

A complex number $z = a + bi$ can be written in **trigonometric form**:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arctan\frac{b}{a}$.

Euler's formula and identity

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

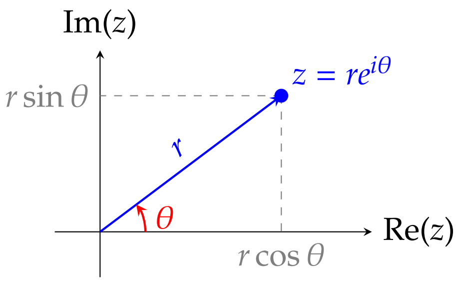
converts the trigonometric form of z into **exponential form**

$$z = re^{i\theta}.$$

For $z = -1$ this gives **Euler's identity**

$$e^{i\pi} + 1 = 0,$$

which is the only simple formula in mathematics that contains the five major constants $0, 1, i, \pi, e$.



Operations in exponential form

Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be complex numbers in exponential form.

$$\begin{aligned}
\textbf{Multiplication:}\quad & z \cdot w = rs\, e^{i(\theta+\phi)} \\
\textbf{Division:}\quad & \frac{z}{w} = \frac{r}{s}\, e^{i(\theta-\phi)}, \quad w \neq 0 \\
\textbf{Inverse:}\quad & z^{-1} = \frac{1}{r}\, e^{-i\theta}, \quad z \neq 0 \\
\textbf{Conjugate:}\quad & \bar{z} = r\, e^{-i\theta}
\end{aligned}$$

Example 3.4: Let $z = 2e^{i\pi/3}$ and $w = 3e^{i\pi/6}$. Then:

$$\begin{aligned}
z \cdot w &= (2 \cdot 3)e^{i(\pi/3+\pi/6)} = 6e^{i\pi/2} = 6i \\
\frac{z}{w} &= \frac{2}{3}e^{i(\pi/3-\pi/6)} = \frac{2}{3}e^{i\pi/6} = \frac{2}{3}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \frac{\sqrt{3}}{3} + \frac{1}{3}i \\
z^{-1} &= \frac{1}{2}e^{-i\pi/3} = \frac{1}{2}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = \frac{1}{4} - i\frac{\sqrt{3}}{4} \\
\bar{z} &= 2e^{-i\pi/3} = 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = 1 - i\sqrt{3}
\end{aligned}$$

Example 3.5: Convert $z = 1 + i$ to exponential form:

$$\begin{aligned}
|z| &= \sqrt{1^2 + 1^2} = \sqrt{2} \\
\arg(z) &= \frac{\pi}{4} \\
z &= \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}e^{i\pi/4}
\end{aligned}$$

Example 3.6: Convert $z = -2 + 2i\sqrt{3}$ to exponential form:

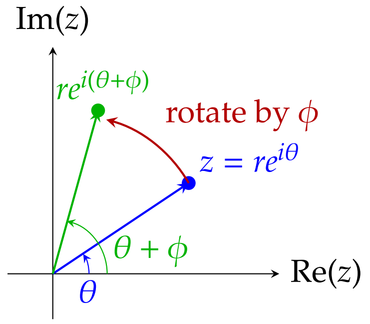
$$\begin{aligned}
|z| &= \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4 \\
\arg(z) &= \pi - \arctan\left(\frac{2\sqrt{3}}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \\
z &= 4e^{i2\pi/3} = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 4(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -2 + 2i\sqrt{3}
\end{aligned}$$

Rotation by multiplication

Multiplying a complex number by $e^{i\phi}$ rotates it counterclockwise by angle ϕ around the origin. Let $z = re^{i\theta}$ and consider multiplication by $e^{i\phi}$:

$$z \cdot e^{i\phi} = re^{i\theta} \cdot e^{i\phi} = re^{i(\theta+\phi)}$$

The number z with argument θ is rotated to a new complex number with argument $\theta + \phi$, while the modulus r remains unchanged.



This geometric interpretation makes complex multiplication particularly useful in applications involving rotations, such as electrical engineering and computer graphics.