2.6.2 Polynomials and rational functions

Polynomial functions

A **polynomial function** is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are constants and $a_n \neq 0$. The number n is called the **degree** of the polynomial, denoted $\deg(p) = n$. The domain is $\mathrm{dom}(p) = \mathbb{R}$.

Example 2.8:

1. Constant functions: $p(x) = c$ (degree 0).
2. Linear functions: $p(x) = ax + b$ with $a \neq 0$ (degree 1).
3. Quadratic functions: $p(x) = ax^2 + bx + c$ with $a \neq 0$ (degree 2).
4. Cubic functions: $p(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ (degree 3).

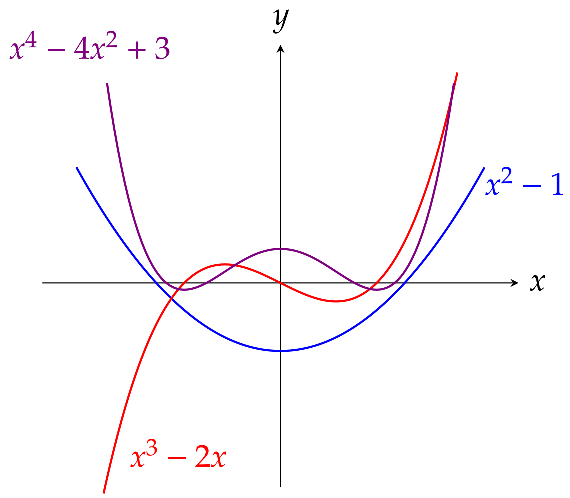


Figure 2.13: Examples of polynomial functions

Properties of polynomials

- The sum and product of two polynomials is a polynomial.
- If p has degree n, then p has at most n real roots (zeros).

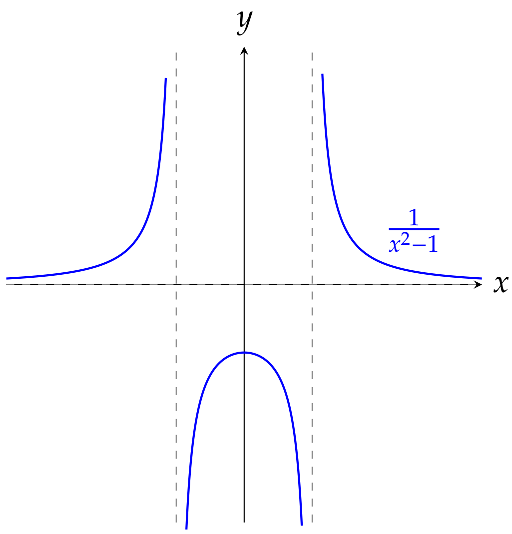
Rational functions

A **rational function** is a quotient of two polynomials:

$$r(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials with $q \not\equiv 0$. If p and q have no common factors then the domain is

$$\text{dom}(r) = \{x \in \mathbb{R} \mid q(x) \neq 0\}.$$



2.6.3 Exponential and logarithmic functions

The exponential function

For $a > 0$, $a \neq 1$, the **exponential function with base** a is

$$f(x) = a^x$$

with $\text{dom}(f) = \mathbb{R}$ and $\text{im}(f) = (0, +\infty)$. The most important case is $a = e$, where $e \approx 2.71828\ldots$ is Euler's number. We write

$$\exp(x) = e^x.$$

Properties of exponential functions

For $a > 0, a \neq 1$:

- a^x is strictly increasing if $a > 1$ and strictly decreasing if $0 < a < 1$.
- $a^0 = 1$ for all $a > 0$.
- $a^x > 0$ for all $x \in \mathbb{R}$.

Algebraic rules for exponentials

For $a, b > 0$ and $x, y \in \mathbb{R}$:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y} \\ \frac{a^x}{a^y} &= a^{x-y} \\ (a^x)^y &= a^{xy} \\ a^x b^x &= (ab)^x \\ a^{-x} &= \frac{1}{a^x} \end{aligned}$$

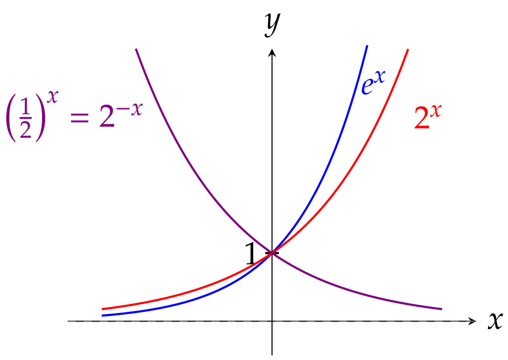


Figure 2.14: Exponential functions with different bases

The logarithmic function

For $a > 0, a \neq 1$, the **logarithm with base** a is the inverse of a^x:

$$f(x) = \log_a(x)$$

with $\mathrm{dom}(f) = (0, +\infty)$ and $\mathrm{im}(f) = \mathbb{R}$. The most important case is $a = e$, called the **natural logarithm**:

$$\ln(x) = \log_e(x).$$

Properties of logarithms

For $a > 0$, $a \neq 1$:

- $\log_a(x)$ is strictly increasing if $a > 1$ and strictly decreasing if $0 < a < 1$.
- $\log_a(1) = 0$ and $\log_a(a) = 1$ for all $a > 0$, $a \neq 1$.
- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$ and $a^{\log_a(x)} = x$ for all $x > 0$.

Algebraic rules for logarithms

For $a > 0$, $a \neq 1$, and $x, y > 0$:

$$\begin{aligned}
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y) \\
\log_a(x^r) &= r\log_a(x) \qquad \text{for } r \in \mathbb{R} \\
\log_a(x) &= \frac{\log_b(x)}{\log_b(a)} \qquad \text{(change of base)}
\end{aligned}$$

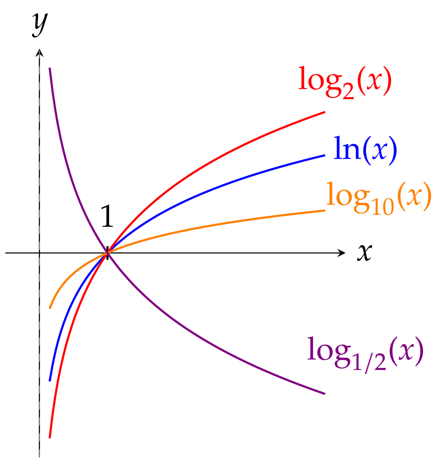


Figure 2.15: Logarithmic functions with different bases

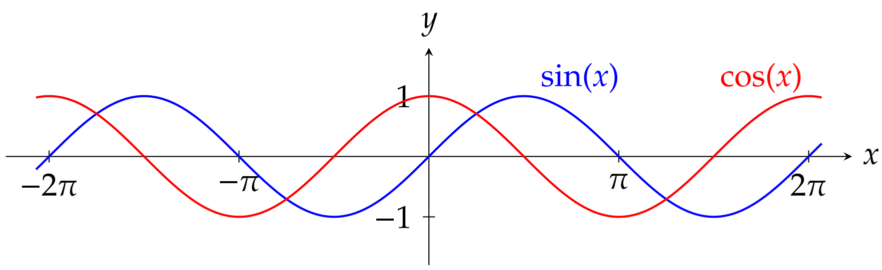
2.6.4 Trigonometric functions and their inverses

Sine and cosine

The **sine** and **cosine** functions are defined as

$$f(x) = \sin(x), \qquad g(x) = \cos(x)$$

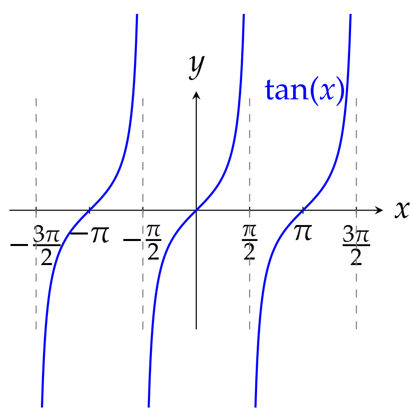
with $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbb{R}$ and $\mathrm{im}(f) = \mathrm{im}(g) = [-1, 1]$. Both functions are periodic with period 2π.

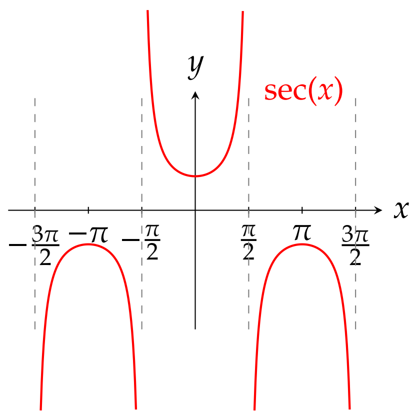


Other trigonometric functions

The other trigonometric functions are defined in terms of sine and cosine:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \mathrm{dom}(\tan) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \;\middle|\; k \in \mathbb{Z} \right\}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \mathrm{dom}(\cot) = \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \}$$
$$\sec(x) = \frac{1}{\cos(x)} \qquad \mathrm{dom}(\sec) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \;\middle|\; k \in \mathbb{Z} \right\}$$
$$\csc(x) = \frac{1}{\sin(x)} \qquad \mathrm{dom}(\csc) = \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \}$$





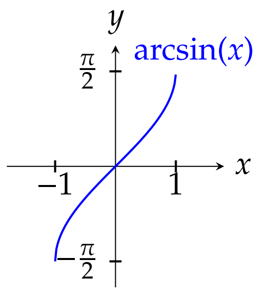
Fundamental trigonometric identities

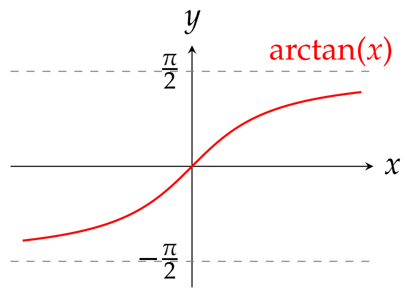
$$\begin{aligned}
\sin^2(\alpha) + \cos^2(\alpha) &= 1 \\
\tan^2(\alpha) + 1 &= \sec^2(\alpha) \\
1 + \cot^2(\alpha) &= \csc^2(\alpha) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) \\
\sin(-\alpha) &= -\sin(\alpha) \qquad \text{(sine is odd)} \\
\cos(-\alpha) &= \cos(\alpha) \qquad \text{(cosine is even)} \\
\sin(\alpha) - \sin(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\
\cos(\alpha) - \cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

Inverse trigonometric functions

The inverse trigonometric functions are defined by restricting the domains appropriately:

- $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse of sin restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- $\arccos : [-1, 1] \to [0, \pi]$ is the inverse of cos restricted to $[0, \pi]$.
- $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is the inverse of tan restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.





Here are some notable values of the trigonometric functions:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-	0