where the function Id is defined as follows:

The identity function

Id is the identity function that can be defined on any subset:

$$\mathrm{Id}_A(x) = x, \qquad \forall x \in A.$$

Here A could be a subset of any set.

Proposition 2.3: Let $f : X \to Y$ and $g : Y \to Z$ both be one-to-one functions. Then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. First, we show that $g \circ f$ is one-to-one, so its inverse exists. Let $x_1, x_2 \in X$ with $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one, $f(x_1) = f(x_2)$. Since f is one-to-one, $x_1 = x_2$. Thus $g \circ f$ is one-to-one.

Now we show $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Let $z \in \mathrm{dom}((g \circ f)^{-1}) = \mathrm{im}(g \circ f)$. Then there exists a unique $x \in X$ such that $(g \circ f)(x) = z$, and $(g \circ f)^{-1}(z) = x$. We have $g(f(x)) = z$, so $f(x) = g^{-1}(z)$, and thus $x = f^{-1}(g^{-1}(z)) = (f^{-1} \circ g^{-1})(z)$. Therefore $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ for all $z \in \mathrm{dom}((g \circ f)^{-1})$. □

Lemma 2.4: Let $f, g : \mathbb{R} \to \mathbb{R}$. Then:

$$\begin{aligned} f, g \text{ are both monotone increasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are both monotone decreasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are monotone of different kinds} \quad &\Rightarrow \quad g \circ f \text{ is monotone decreasing.} \end{aligned}$$

Proof. Exercise. □

Translations, rescalings and reflections

Here we discuss three types of simple functions that might often appear as part of a composition of functions.

Translations are an important family of functions: they simply 'move' the variable x by some fixed amount. Here is the simple definition. Given a fixed $c \in \mathbb{R}$, define the **translation by** c, the function $t_c : \mathbb{R} \to \mathbb{R}$, as

$$t_c(x) = x + c.$$

Here x is the variable, as always, and the number c is normally called a *parameter*. Given some function $f : \mathbb{R} \to \mathbb{R}$, we see that

$$\begin{aligned} (f \circ t_c)(x) &= f(t_c(x)) = f(x + c), \\ (t_c \circ f)(x) &= t_c(f(x)) = f(x) + c. \end{aligned}$$

The first function is a shift of the graph of f to the left by c (if $c < 0$ then the shift is to the right). The second function is a shift of the graph of f up by c (if $c < 0$ then the shift is down).

Rescalings 'squeeze' or 'stretch' a function horizontally. Here's the definition. Given a fixed $c > 0$, define the **scaling by** c, the function $s_c : \mathbb{R} \to \mathbb{R}$, as

$$s_c(x) = cx.$$

Composition with some function $f : \mathbb{R} \to \mathbb{R}$ gives

$$(f \circ s_c)(x) = f(s_c(x)) = f(cx).$$

If $0 < c < 1$ then the graph of f is 'stretched' by a factor of $\frac{1}{c}$, whereas if $c > 1$ then the graph of f is 'squeezed' by a factor of c, see Figure 2.8.

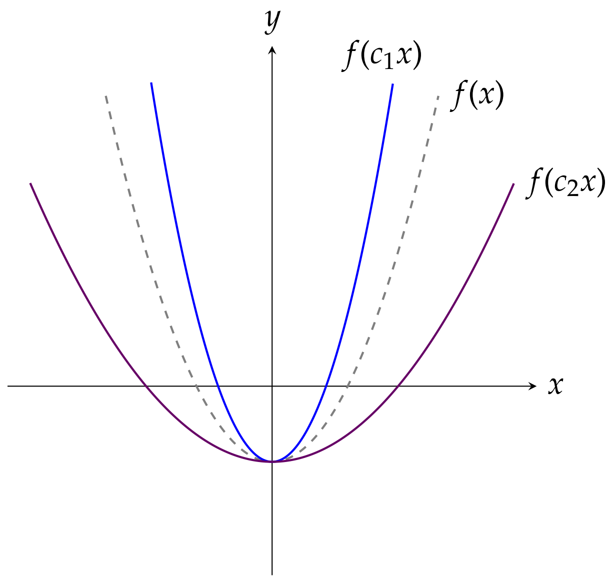

Figure 2.8: Rescalings of a function $f(x)$ by $c_1 > 1$ and $0 < c_2 < 1$.

We can also rescale a function vertically. Given a fixed $c > 0$, define

$$(s_c \circ f)(x) = s_c(f(x)) = cf(x).$$

A constant $c > 1$ will 'stretch' the graph of the function along the y-axis, and $0 < c < 1$ will 'squeeze' the graph of the function.

Reflection of a function 'flips' the graph of a function along the y-axis. Define the **reflection function** $r : \mathbb{R} \to \mathbb{R}$ as

$$r(x) = -x.$$

Then

$$(f \circ r)(x) = f(r(x)) = f(-x).$$

Switching the order of composition of the two functions will result in flipping the graph along the x-axis:

$$(r \circ f)(x) = r(f(x)) = -f(x).$$

2.6 Elementary functions

Even, odd and periodic functions

Even and odd functions

Let $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ and assume that if $x \in \mathrm{dom}(f)$ then also $-x \in \mathrm{dom}(f)$. We say that f is **even** if

$$f(x) = f(-x), \qquad \forall x \in \mathrm{dom}(f).$$

We say that f is **odd** if

$$f(x) = -f(-x), \qquad \forall x \in \mathrm{dom}(f).$$

Periodic functions

A function $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ is said to have **period** $p > 0$ if $\forall x \in \mathrm{dom}(f)$, $\{x + np \mid n \in \mathbb{Z}\} \subseteq \mathrm{dom}(f)$ and if

$$f(x) = f(x + np), \qquad \forall x \in \mathrm{dom}(f),\ \forall n \in \mathbb{Z}.$$

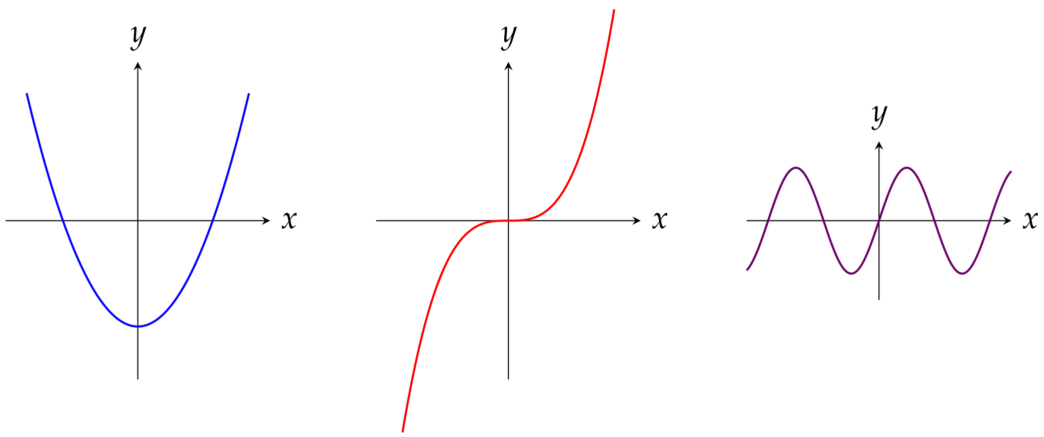

Figure 2.9: Examples of an even function (left), an odd function (middle), and a periodic function (right).

2.6.1 Powers

Non-negative integer powers

Power functions

A (non-negative integer) **power function** is a function of the form

$$f(x) = x^n, \qquad n \in \mathbb{N}.$$

The domain is $\mathrm{dom}(f) = \mathbb{R}$ for all $n \in \mathbb{N}$.

Properties

- If n is even, then $f(x) = x^n$ is an even function: $f(-x) = (-x)^n = x^n = f(x)$.
- If n is odd, then $f(x) = x^n$ is an odd function: $f(-x) = (-x)^n = -x^n = -f(x)$.
- For $n \geq 1$, $f(x) = x^n$ is strictly increasing on $[0, +\infty)$.
- For even $n \geq 2$, $f(x) = x^n$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.
- For odd $n \geq 1$, $f(x) = x^n$ is strictly increasing on $\mathbb{R}$.

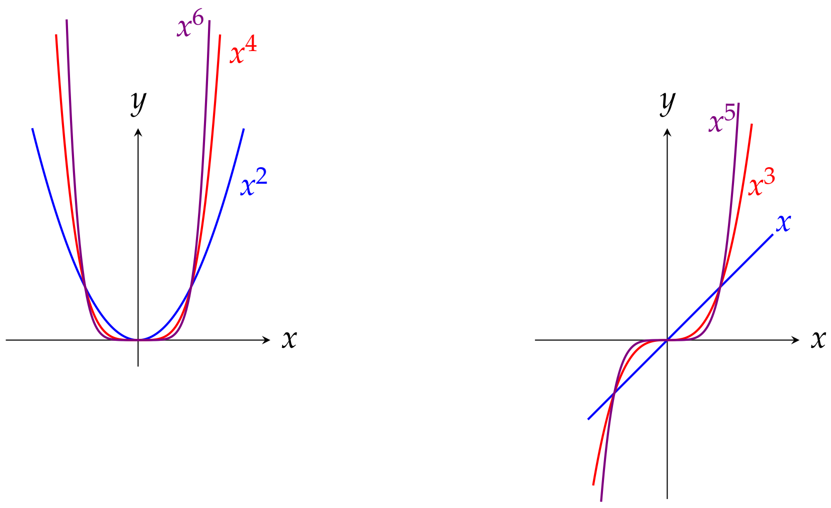

Figure 2.10: Even powers (left) and odd powers (right)

Positive rational powers

qth root

The power function $f(x) = x^{1/q}$ where $q \in \{2, 3, \dots\}$ is called the **qth root of x** and is denoted

$$f(x) = \sqrt[q]{x}, \qquad \operatorname{dom}(f) = \begin{cases} \mathbb{R} & q \text{ is odd} \\ [0, +\infty) & q \text{ is even} \end{cases}$$

It is the inverse function of $y = x^q$.

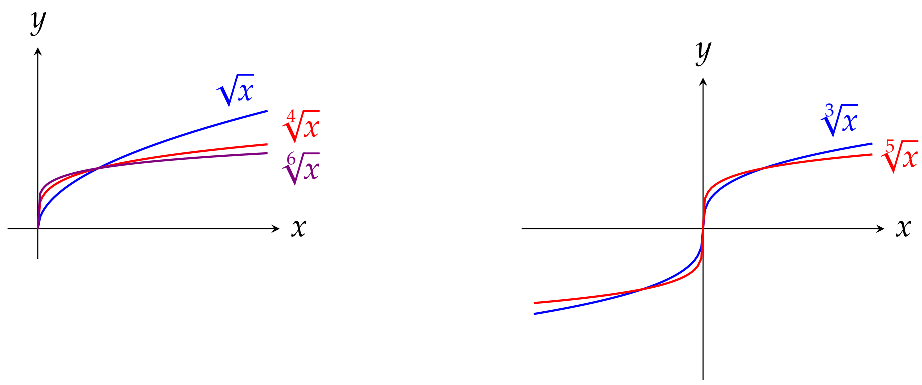

Figure 2.11: Roots with even index (left) and odd index (right)

Positive rational power

Let $p, q \in \mathbb{N}$ have no common divisors. A power function x^r with $r = \frac{p}{q}$ is defined as follows:

$$f(x) = x^r = x^{p/q} = (x^p)^{1/q} = \sqrt[q]{x^p}, \qquad \operatorname{dom}(f) = \begin{cases} \mathbb{R} & q \text{ is odd} \\ [0, +\infty) & q \text{ is even} \end{cases}$$

Positive irrational powers

It is not obvious how to define an irrational power. For instance, what is the meaning of

$$\pi^{\sqrt{2}} = ?$$

Since we know that $\sqrt{2} = 1.414213562\cdots$ we can define the number $\pi^{\sqrt{2}}$ to be the number that we approach by taking better and better approximations of $\sqrt{2}$:

$$\begin{aligned}&\pi^{1.4}\\&\pi^{1.41}\\&\pi^{1.414}\\&\pi^{1.4142}\\&\pi^{1.41421}\\&\pi^{1.414213}\\&\vdots\\&\pi^{\sqrt{2}}\end{aligned}$$

However this is not *a priori* simple. We skip this important problem in this course.

Positive irrational power

A power function x^s with $s \in \mathbb{R}_+ \setminus \mathbb{Q}$ is defined as the 'limit' of x^r where $r \in \mathbb{Q}$ and r approaches s. The **domain is** $[0, +\infty)$.

Negative powers

Negative power functions

- For $n \in \mathbb{N}_+$ the negative power function is

$$f(x) = x^{-n} = \frac{1}{x^n}, \qquad \mathrm{dom}(f) = \mathbb{R} \setminus \{0\}.$$

- For $r = \frac{p}{q} \in \mathbb{Q}$ (with $p, q \in \mathbb{N}_+$ having no common divisors)

$$f(x) = x^{-r} = \frac{1}{\sqrt[q]{x^p}}, \qquad \mathrm{dom}(f) = \begin{cases} \mathbb{R} \setminus \{0\} & q \text{ is odd} \\ (0, +\infty) & q \text{ is even} \end{cases}$$

- For $s \in \mathbb{R}_+ \setminus \mathbb{Q}$ the negative power function is

$$f(x) = x^{-s} = \frac{1}{x^s}, \qquad \mathrm{dom}(f) = (0, +\infty).$$

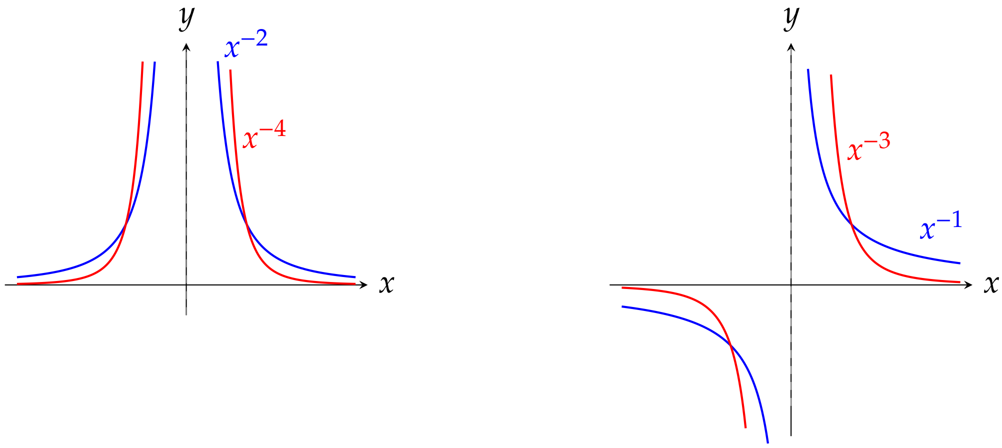

Figure 2.12: Negative powers with even exponent (left) and odd exponent (right)

Properties of negative integer powers

- For $n \in \mathbb{N}_+$, $f(x) = x^{-n}$ is strictly decreasing on $(0, +\infty)$ and on $(-\infty, 0)$.
- If n is even, $f(x) = x^{-n}$ is an even function.
- If n is odd, $f(x) = x^{-n}$ is an odd function.

Properties of real powers

For $x > 0$ and $r, s \in \mathbb{R}$:

- $f(x) = x^r$ is strictly increasing on $(0, +\infty)$ if $r > 0$.
- $x^r \cdot x^s = x^{r+s}$
- $\frac{x^r}{x^s} = x^{r-s}$
- $(x^r)^s = x^{rs}$
- For $0 < r < 1$, the function x^r is concave (bends downward).
- For $r > 1$, the function x^r is convex (bends upward).