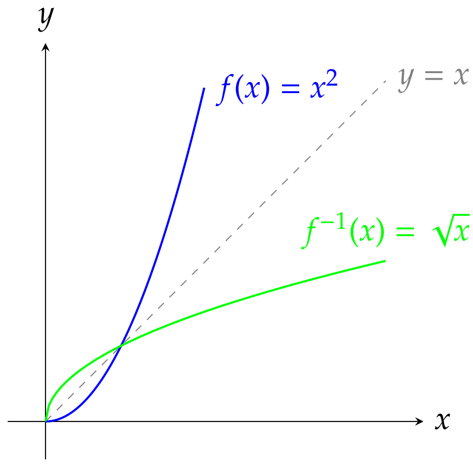

Figure 2.4: $f(x) = x^2$ and its inverse $f^{-1}(x) = \sqrt{x}$, defined on $x \geq 0$, are mirror images with respect to the line $x = y$

Sometimes we want to only look at part of the domain of a function. For example, in the example above, we looked at $f(x) = x^2$ only for $x \geq 0$, so that we could look at its inverse. Otherwise, if we had looked at $x \in \mathbb{R}$, then the preimage of any $y \geq 0$ is $\{+\sqrt{y}, -\sqrt{y}\}$ – i.e., there is no inverse function. What we did was to *restrict* $f(x) = x^2$ to $x \geq 0$:

Restriction of a function

Let $f : X \to Y$ be a function. Let $A \subseteq \text{dom}(f)$ be a subset of the domain of f. The restriction of f to A is a 'new' function $f|_A$ that is defined only on A, where it is identical to f:

$$f|_A : A \to Y \qquad \text{defined as} \qquad f|_A(x) = f(x), \quad \forall x \in A.$$

In Figure 2.4,the blue graph is the graph of the restriction of x^2 to $A = \{x \in \mathbb{R} \mid x \geq 0\}$.

2.4 Monotone functions and sequences

Functions that always increase/decrease are of particular interest because they might have important applications. For example, when you study *thermodynamics* you will see that **entropy** is a monotone increasing function of time, meaning that our world always become more disorganized.

Monotone functions $f : \mathbb{R} \to \mathbb{R}$

Monotone functions $f : \mathbb{R} \to \mathbb{R}$

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **(monotonically) increasing on** $I \subseteq \mathrm{dom}(f)$ if for every $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) \le f(x_2)$:

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \le f(x_2).$$

The function f is said to be **strictly increasing on** I if

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

Similarly, f is said to be **(monotonically) decreasing on** I if for every $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) \ge f(x_2)$:

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \ge f(x_2).$$

The function f is said to be **strictly decreasing on** I if

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

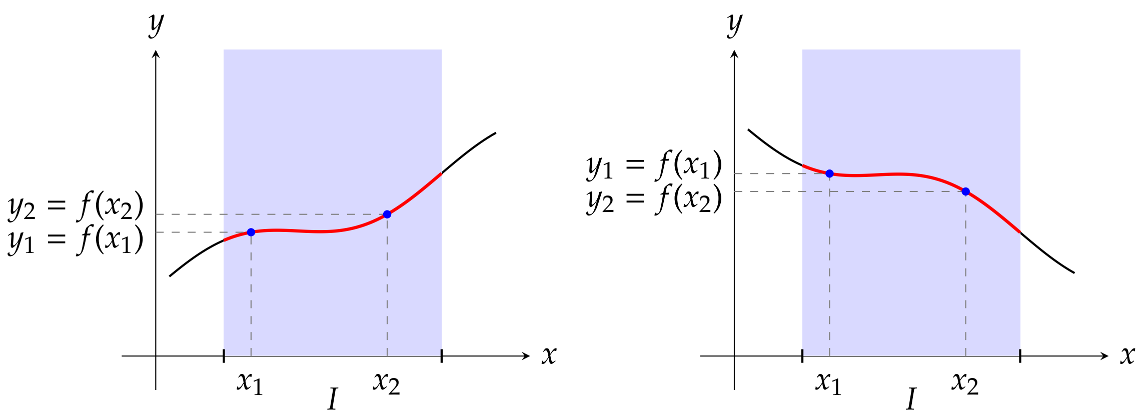

Figure 2.5: An increasing function (left) and a decreasing function (right)

Note that monotonically increasing/decreasing functions are allowed to have slope 0 and even to remain constant. So a constant function $f(x) = c$ is (trivially) both monotone increasing and monotone decreasing. Step functions such as $f(x) = \lceil x \rceil$ or $f(x) = \lfloor x \rfloor$ are monotonically increasing (but not decreasing).

Example 2.4:

1. $f(x) = c$ is monotonically increasing and decreasing.
2. $f(x) = x^2$ is neither increasing nor decreasing on $\mathbb{R}$.
3. $f(x) = x^2$ is strictly increasing on $[0, +\infty)$.
4. $f(x) = x^2$ is strictly decreasing on $(-\infty, 0]$.

5. $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.

6. $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$ is monotonically increasing on $\mathbb{R}$, and it is *strictly* increasing on $[0, +\infty)$.

Proposition 2.1: A function that is strictly increasing/decreasing on its domain is injective (one-to-one).

Proof. Consider first the case that $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing on $\mathrm{dom}(f)$. Let $x_1, x_2 \in \mathrm{dom}(f)$ with $x_1 \neq x_2$. Without loss of generality $x_1 < x_2$, so that $f(x_1) < f(x_2)$. In particular, $f(x_1) \neq f(x_2)$, so that f is injective. The case of a strictly decreasing function follows the same idea of proof. □

★★★ The converse statement – i.e. that an injective function is strictly monotone – is not true, see Figure 2.6 for a counterexample.

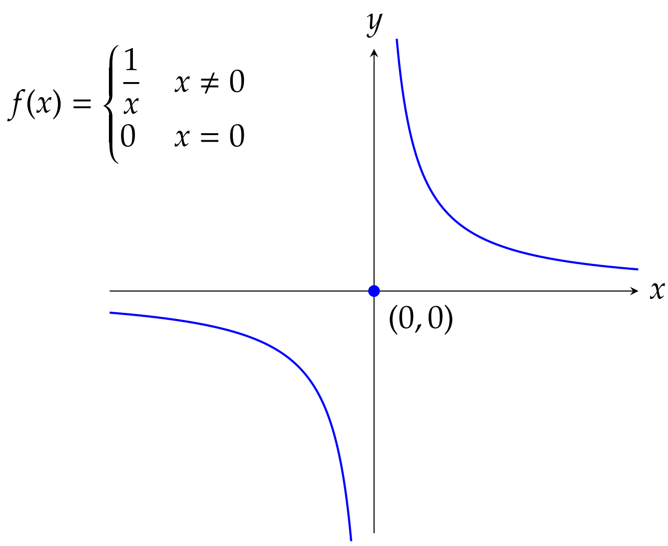

Figure 2.6: A one-to-one function that is neither increasing nor decreasing

Lemma 2.2: Let $f, g : \mathbb{R} \to \mathbb{R}$ be monotonically increasing on some $A \subseteq \mathbb{R}$. Then $f + g$ is also monotonically increasing on A. If either f or g are *strictly* increasing on A, then so is $f + g$. The same statements hold if we replace everywhere the word 'increasing' with the word 'decreasing'.

Proof. Exercise. □

Monotone sequences

A sequence a_n is said to be **(monotonically) increasing on** $\{N, N+1, \dots\}$ if

$$\forall n \geq N, \qquad a_n \leq a_{n+1}.$$

A sequence a_n is said to be **strictly increasing on** $\{N, N+1, \dots\}$ if

$$\forall n \geq N, \qquad a_n < a_{n+1}.$$

A sequence a_n is said to be **(monotonically) decreasing on** $\{N, N+1, \dots\}$ if

$$\forall n \geq N, \qquad a_n \geq a_{n+1}.$$

A sequence a_n is said to be **strictly decreasing on** $\{N, N+1, \dots\}$ if

$$\forall n \geq N, \qquad a_n > a_{n+1}.$$

Example 2.5:

1. The sequence $a_n = \frac{1}{n}$, $n \in \mathbb{N}_+$, is strictly decreasing.
2. The sequence $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}$ is strictly increasing.
3. The sequence $a_n = (-1)^n$, $n \in \mathbb{N}$ is neither increasing nor decreasing.

2.5 Composition of functions

The composition of functions – i.e. the application of two (or more) functions successively – is something that often comes up in mathematics and its applications.

Example 2.6 (Taxi fare)**:** Suppose that the fare for riding a taxi is made of a flat fee of 3 Euros plus twice the distance travelled (in kilometers). So the fee for riding x kilometers is:

$$f(x) = 2x + 3.$$

Now, suppose that a card payment carries a 5% surcharge of the total fare:

$$g(y) = 1.05y.$$

So, if we travel x kilometers, and want to pay by card, the total amount to pay is:

$$g(f(x)) = 1.05(2x + 3) = 2.1x + 3.15$$

This is a composition of functions.

Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be two functions. The **composition of f and g** is a new function $h : X \to Z$ defined as

$$h(x) = g(f(x)).$$

It is denoted by $h = g \circ f$ so we can also write $h(x) = (g \circ f)(x)$.

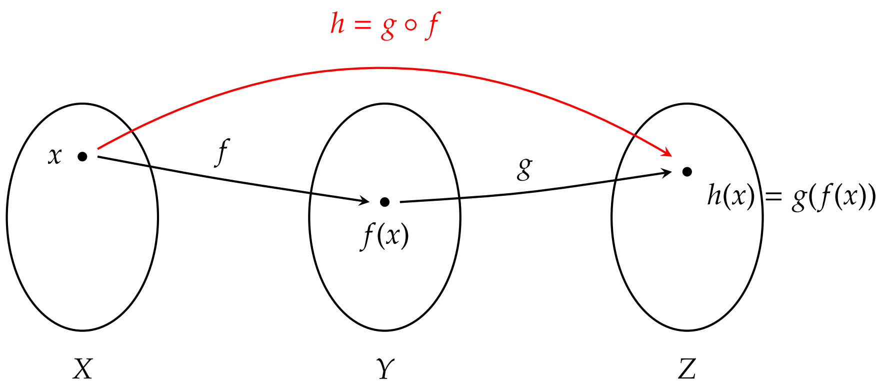

Figure 2.7: Composition of functions $h(x) = g(f(x))$

The domain of h is defined as follows:

$$x \in \mathrm{dom}(h) \quad \Leftrightarrow \quad x \in \mathrm{dom}(f) \quad \text{and} \quad f(x) \in \mathrm{dom}(g).$$

Example 2.7:

1. If $f(x) = x^2$, $g(y) = y - 3$, then $h(x) = g(f(x)) = x^2 - 3$, and $\mathrm{dom}(h) = \mathrm{dom}(f)$.
2. If $f(x) = e^x$, $g(y) = -y$, then $h(x) = g(f(x)) = -e^x$, and $\mathrm{dom}(h) = \mathrm{dom}(f)$.
3. If $f(x) = -x$, $g(y) = e^y$, then $h(x) = g(f(x)) = e^{-x}$, and $\mathrm{dom}(h) = \mathrm{dom}(f)$.
4. If $f(x) = \sqrt{x}$, $g(y) = y^2$, then $h(x) = g(f(x)) = (\sqrt{x})^2 = x$, and $\mathrm{dom}(h) = \mathrm{dom}(f) = [0, +\infty)$.
5. If $f(x) = x^2$, $g(y) = \sqrt{y}$, then $h(x) = g(f(x)) = \sqrt{x^2} = |x|$, and $\mathrm{dom}(h) = \mathrm{dom}(f) = \mathbb{R}$.
6. If $f(x) = \frac{1}{x}$, $g(y) = \sin y$, then $h(x) = g(f(x)) = \sin\frac{1}{x}$, and $\mathrm{dom}(h) = \mathrm{dom}(f) = \mathbb{R}\backslash\{0\}$.
7. If $f(x) = \sin x$, $g(y) = \frac{1}{y}$, then $h(x) = g(f(x)) = \frac{1}{\sin x}$, and $\mathrm{dom}(h) \neq \mathrm{dom}(f)$. In this case $\mathrm{dom}(h)$ is all $x \in \mathbb{R}$ s.t. $\sin x \neq 0$.

These examples show us that the **composition of functions is not a commutative operation**:

$$f \circ g \neq g \circ f.$$

We can also see that a function and its inverse ‘cancel’ one another. More precisely, if f is one-to-one (and therefore f^{-1} exists) then

$$f \circ f^{-1} = \mathrm{Id}_{\mathrm{dom}(f^{-1})} = \mathrm{Id}_{\mathrm{im}(f)} \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{Id}_{\mathrm{dom}(f)}$$