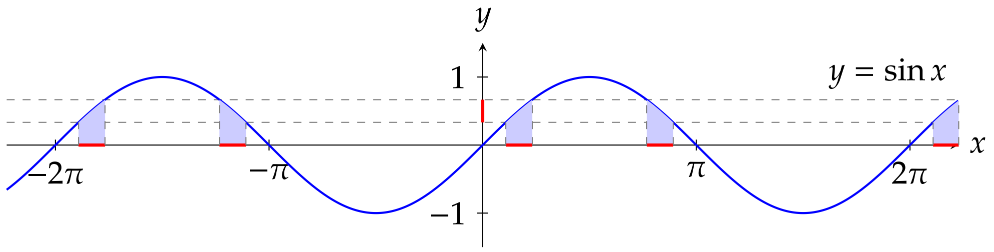


We can now talk about the supremum, infimum, maximum and minimum of the image of various sets under a real-valued function f:

Supremum and infimum of a real-valued function

Let $f : X \to \mathbb{R}$ be a real-valued function. Let $A \subseteq \text{dom}(f)$. The **supremum of** f **on** A is the supremum of the image of A under f:

$$\sup_A f = \sup_{x \in A} f(x) = \sup\{f(x) \mid x \in A\}.$$

Similarly, the **infimum of** f **on** A is the infimum of the image of A under f:

$$\inf_A f = \inf_{x \in A} f(x) = \inf\{f(x) \mid x \in A\}.$$

As we have already seen, the supremum can be an element of $\mathbb{R} \cup \{+\infty\}$ and the infimum can be an element of $\{-\infty\} \cup \mathbb{R}$.

Boundedness of a real-valued function

If $\sup_{x \in A} f(x) < +\infty$ (i.e. it is a real number), we say that f is **bounded from above on** A. If $\inf_{x \in A} f(x) > -\infty$ (i.e. it is a real number), we say that f is **bounded from below on** A. If f is bounded from above and below on A, we say that it is **bounded on** A.

Maximum and minimum of a real-valued function

If $\sup_{x \in A} f(x) < +\infty$ and it belongs to $f(A)$ then it is the **maximum of** f **on** A. It is denoted

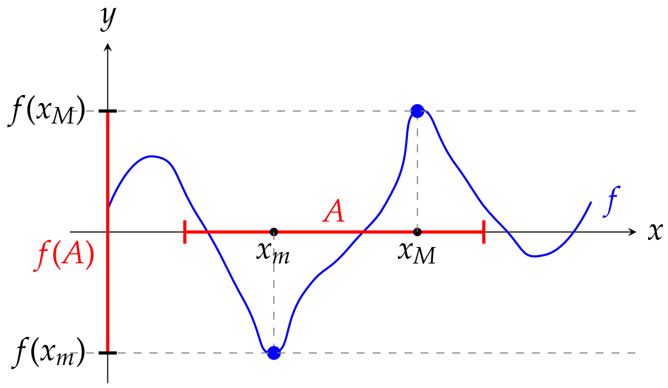
$$\max_A f \qquad \text{or} \qquad \max_{x \in A} f(x).$$

If $\inf_{x \in A} f(x) > -\infty$ and it belongs to $f(A)$ then it is the **minimum of** f **on** A. It is denoted

$$\min_A f \qquad \text{or} \qquad \min_{x \in A} f(x).$$

Since the minimum and the maximum of f on A belong to $f(A)$, there exist $x_m \in A$ and $x_M \in A$ such that

$$f(x_M) = \max_A f \qquad \text{and} \qquad f(x_m) = \min_A f.$$



Example 2.3: 1. For $\sin x$, we have

$$\max_{x\in\mathbb{R}}(\sin x) = 1 \qquad \text{and} \qquad \min_{x\in\mathbb{R}}(\sin x) = -1.$$

2. For x^2 we have

$$\sup_{x\in\mathbb{R}}(x^2) = +\infty \qquad \text{and} \qquad \min_{x\in\mathbb{R}}(x^2) = 0.$$

If $A = [-10, -3)$ then

$$\max_{x\in A}(x^2) = 100 \qquad \text{and} \qquad \inf_{x\in A}(x^2) = 9.$$

Note that in this last case, the infimum is not achieved, so there's no minimum on A.

2.3 Surjectivity, injectivity, and invertibility

Let us define some important *global* properties of functions, i.e. properties that tell us something about the function as a whole.

Definitions

Let $f : X \to Y$.

- We say that f is **surjective** (or **onto**) if $\mathrm{im}(f) = Y$.
- We say that f is **injective** (or $1-1$, **one-to-one**) if for every $y \in Y$, the subset $f^{-1}(y) \subseteq X$ contains *at most* one element.
- If f is both surjective and injective, it is called a **bijection** (or a **bijective function**).

Let us try to understand these concepts. Surjectivity means that for every $y \in Y$ there exists (at least) one $x \in X$ such that $f(x) = y$. See Figure 2.1.

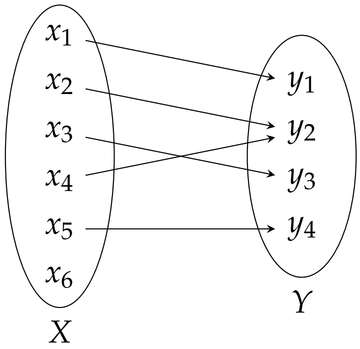


Figure 2.1: A surjective (*onto*) function

Injectivity means that every $y \in Y$ has at most one pre-image. So for any $y \in Y$, either there is no $x \in X$ such that $f(x) = y$ or there is one (and no more) $x \in X$ such that $f(x) = y$. Equivalently, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. See Figure 2.2.

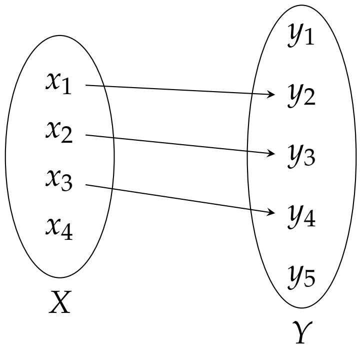


Figure 2.2: An injective (*1-1*) function

Finally, a bijective function f associates to every $x \in X$ exactly one $y \in Y$, and vice versa. See Figure 2.3. In this case we say that the sets X **and** Y **are in one-to-one (*1-1*) correspondence** under f.

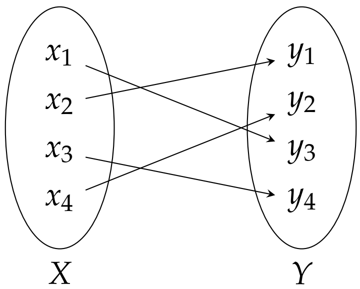


Figure 2.3: A bijective function

The case of a function $f : \mathbb{R} \to \mathbb{R}$

When $f : \mathbb{R} \to \mathbb{R}$ the aforementioned properties can be seen by looking at its graph in $\mathbb{R}^2$:

- f is *surjective* if its graph intersects any horizontal line <u>at least</u> once;
- f is *injective* if its graph intersects any horizontal line <u>at most</u> once;
- f is *bijective* if its graph intersects every horizontal line <u>exactly</u> once.

We observe that the arrows of an injective (1-1) function can be reversed to obtain a function from Y to X. So we define:

Inverse function

Let $f : X \to Y$ be one-to-one. We define its **inverse** $f^{-1} : \mathrm{im}(f) \subseteq Y \to X$ as follows:

$$f^{-1}(y) = x \qquad \text{where } x \text{ is the unique element in } X \text{ satisfying } f(x) = y.$$

Therefore any one-to-one function is also **invertible**. Observe that:

$$\mathrm{dom}(f) = \mathrm{im}(f^{-1}) \qquad \text{and} \qquad \mathrm{im}(f) = \mathrm{dom}(f^{-1})$$

Note that there is some abuse of notation[1] here: $f^{-1}(y)$ is defined to be the *element* $x \in X$, while before (see Section 2.2) we defined $f^{-1}(y)$ to be the *subset* $\{x\} \subseteq X$. In the case of a one-to-one function, this can be forgiven, because there is little practical difference between x and $\{x\}$.

The graph of the inverse function

Recall that the graph of $f : \mathrm{dom}(f) \subseteq X \to Y$ is defined as

$$\Gamma(f) = \{(x, f(x) \in X \times Y \mid x \in \mathrm{dom}(f)\}.$$

This immediately implies that

$$\begin{aligned}\Gamma(f^{-1}) &= \{(y, f^{-1}(y)) \in Y \times X \mid y \in \mathrm{dom}(f^{-1})\} \\ &= \{(f(x), x) \in Y \times X \mid x \in \mathrm{dom}(f)\}.\end{aligned}$$

Comparing the two expressions for $\Gamma(f)$ and for $\Gamma(f^{-1})$, one can see that the graph of f^{-1} is obtained by mirroring the graph of f along the $x = y$ line. We can clearly see this for $f(x) = x^2$, $x \geq 0$, and its inverse $f^{-1}(x) = \sqrt{x}$, see Figure 2.4.

[1] *"Abuse of notation"* means that our notation is not entirely consistent.

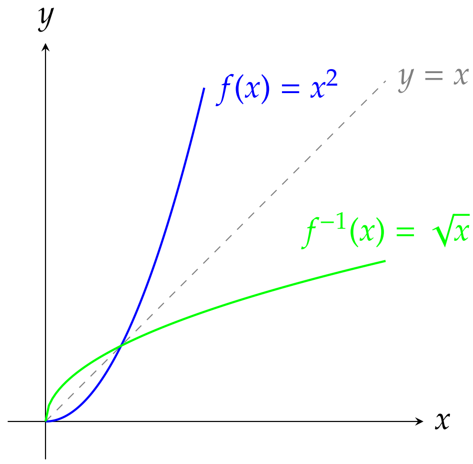


Figure 2.4: $f(x) = x^2$ and its inverse $f^{-1}(x) = \sqrt{x}$, defined on $x \geq 0$, are mirror images with respect to the line $x = y$

Sometimes we want to only look at part of the domain of a function. For example, in the example above, we looked at $f(x) = x^2$ only for $x \geq 0$, so that we could look at its inverse. Otherwise, if we had looked at $x \in \mathbb{R}$, then the preimage of any $y \geq 0$ is $\{+\sqrt{y}, -\sqrt{y}\}$ – i.e., there is no inverse function. What we did was to *restrict* $f(x) = x^2$ to $x \geq 0$:

Restriction of a function

Let $f : X \to Y$ be a function. Let $A \subseteq \text{dom}(f)$ be a subset of the domain of f. The restriction of f to A is a 'new' function $f|_A$ that is defined only on A, where it is identical to f:

$$f|_A : A \to Y \qquad \text{defined as} \qquad f|_A(x) = f(x), \quad \forall x \in A.$$

In Figure 2.4,the blue graph is the graph of the restriction of x^2 to $A = \{x \in \mathbb{R} \mid x \geq 0\}$.

2.4 Monotone functions and sequences

Functions that always increase/decrease are of particular interest because they might have important applications. For example, when you study *thermodynamics* you will see that **entropy** is a monotone increasing function of time, meaning that our world always become more disorganized.