1.6 Factorials and binomial coefficients

Here we briefly discuss notions that arise in mathematical fields such as *Discrete Mathematics*, *Combinatorics* and *Probability*. The most basic problem is as follows. Consider the set X of n students

$$X = \underbrace{\{\text{Andrea}, \text{Marta}, \text{Jim}, \text{Victoria}, \dots, \text{Caterina}\}}_{n \text{ students}}$$

As this is a *set*, there is no importance for the ordering of the students. We could express X also as

$$X = \{\text{Caterina}, \text{Marta}, \text{Victoria}, \text{Andrea}, \dots, \text{Jim.}\}$$

However, the elements of X (the n students) are all distinct, and we often care about their ordering. can ask two natural questions:

1. In how many ways can we order the n students? Above we have seen two examples of how to order (or **permute**) the students. There is a simple formula that gives us the number of *permutations*:

- We start by choosing the first student. There are n students in total, hence we have n students to choose from.
- Next, we want to choose the second student to appear in our ordering. We have already chosen one student, so there are only $n-1$ students left to choose from.
- For the third student we have $n-2$ students to choose from.
- And so on.....
- For the $(n-1)st$ student we have two students to choose from.
- For the nth student we no longer have a choice.

Hence, we find that the number of possible permutations of n elements is

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1.$$

n factorial

Since this is an important formula, it has its own special symbol:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$$

which is called n **factorial**. We also define

$$0! = 1.$$

2. In how many ways can we choose k students of the n students? Repeating the same argument as above, we have

- n options for the first student,
- $n-1$ options for the second student,
- $n-2$ options for the third student,
- and so on....
- $n-(k-1) = n-k+1$ options for the kth student.

So we get

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+2) \cdot (n-k+1).$$

We observe that this expression can be simplified:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{(n-k) \cdot (n-k-1) \cdot (n-k-2) \cdots 2 \cdot 1} = \frac{n!}{(n-k)!}$$

Now we make an important observation: the ordering of the k students *does not matter for us*. So, for example if $k = 2$, there is no difference for us between {Jim, Victoria} and {Victoria, Jim}. So we need to eliminate such repetitions. But these repetitions are precisely the number of possible permutations of k elements, which we have seen: it is $k!$. Hence we need to divide the above formula by this number.

n choose k

The number of possible ways to choose k elements from a total of n elements (where $0 \leq k \leq n$) is called n choose k and is denoted

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This is also called the **binomial coefficient**.

It turns out that these coefficient satisfy certain recursive relations (that is, one can compute $\binom{n}{k}$ from knowledge of $\binom{n-1}{j}$ for all $0 \leq j \leq n-1$). The simple way to visualize this is through *Pascal's triangle*, whose first eight lines look as follows:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

If we let n denote the line number (starting from 0) and k denote the position of a given number (also, starting from 0) within a line, then the number appearing in the triangle is precisely $\binom{n}{k}$. Hence, for instance,

$$1 = \binom{0}{0} = \binom{n}{0} = \binom{n}{n}$$

for all $n \in \mathbb{N}$, and

$$n = \binom{n}{1} = \binom{n}{n-1}$$

for all $n \in \mathbb{N}_+$. Some other specific examples are

$$6 = \binom{4}{2}, \qquad 10 = \binom{5}{2} = \binom{5}{3}, \qquad 35 = \binom{7}{3} = \binom{7}{4}.$$

Exercise 1.1: Can you see the pattern?

Newton's binomial formula

For any $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, there holds **Newton's binomial formula**:

$$(a+b)^n = a^n + na^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + nab^{n-1} + b^n$$

Exercise 1.2: Prove Newton's binomial formula.

Summation notation

Let $n, N \in \mathbb{Z}$ with $N \geq n$. Let $c_n, \ldots, c_N \in \mathbb{R}$. Then the expression

$$\sum_{k=n}^{N} c_k$$

is a concise way to write the sum

$$c_n + \cdots + c_N.$$

The integers n and N are known as the **lower** and **upper limits of summation**, respectively. The **subscripts** $n, \ldots, N$ are called the **indices**. The numbers c_k $(k = n, \ldots, N)$ are called the **summands**. The summation can also be denoted:

$$\sum_{k=n}^{N} = \sum_{k \in \{n, \ldots, N\}}$$

indicating that the summation is over the set of integer indices $\{n, \ldots, N\}$. The symbol k is known as the **summation index** and it is a **dummy variable**: this means that it can be replaced by any other symbol without changing the meaning of the expression:

$$\sum_{k=n}^{N} c_k = \sum_{j=n}^{N} c_j = \sum_{m=n}^{N} c_m = \sum_{\star=n}^{N} c_\star = \sum_{\clubsuit=n}^{N} c_\clubsuit$$

The summation notation allows us to simplify the expression for Newton's binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$