Note that in the above definition, we defined *the* supremum and *the* infimum, implying that these two numbers are unique. *A priori*, that is not obvious (even though it is true). It requires a proof.

Proposition 1.7: For any subset $A \subseteq \mathbb{R}$, there are unique elements $\ell, s \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ such that $\ell = \inf A$ and $s = \sup A$.

Proof. Exercise. *Hint: prove it by contradiction.* □

The supremum and the infimum might not belong to the set!

It is very important to remember that for a subset $A \subseteq \mathbb{R}$, its supremum and its infimum might *not* belong to it. We have seen it with A_4 above: its infimum is 0, yet $0 \notin A_4$. For $A_2 = \mathbb{N}$, the supremum is $+\infty$, which isn't a number, and in particular isn't an element of A_2.

Keeping in mind the preceding comment, in the case that the supremum and/or infimum *do* belong to the set we give them another name:

Maximum and minimum

Let $A \subset \mathbb{R}$ be a subset. If $\sup A \in A$ then we say that the supremum is *attained*, and it is called the **maximum** of A and denoted

$$\max A.$$

Similarly, if $\inf A \in A$ then we say that the infimum is *attained*, and it is called the **minimum** of A and denoted

$$\min A.$$

1.3.3 The cardinality of subsets of $\mathbb{R}$

The *cardinalty* of a set A is a measure of its size. The cardinality of a set containing finitely many elements is simply the number of elements: the cardinality of $A = \{-11, 600, \sqrt{17}\}$ is 3. The cardinality of the set $B = \{\text{Giulia}, \text{Sam}, \text{Amelia}\}$ is also 3. The concept of cardinality becomes more delicate when dealing with sets containing infinitely many elements.

Countable sets

A subset $A \subseteq \mathbb{R}$ is said to be **countable** if it is possible to enumerate all its elements.

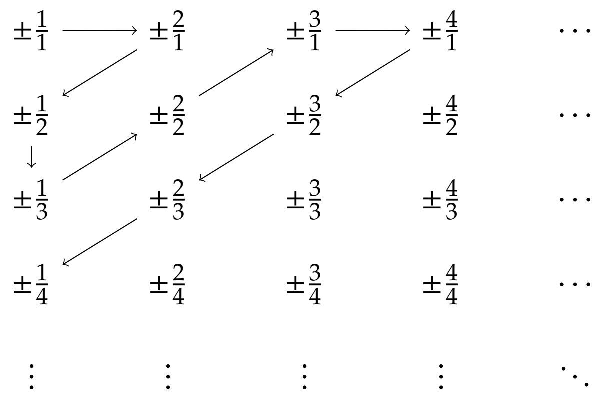
Example 1.5: 1. Any set with finitely many elements is countable.

2. The set $\mathbb{N}_+ = \{1, 2, 3, \dots\}$ is trivially countable: we assign to the element $1 \in \mathbb{N}_+$ the number 1, to the element $2 \in \mathbb{N}_+$ the number 2, and so on.

3. The sets $\mathbb{N}, \mathbb{Z}$ are also countable. The set of even natural numbers is also countable. *Can you prove it?*

Proposition 1.8: The set $\mathbb{Q}$ of rational numbers is countable.

Proof. To prove that a set is countable we need to demonstrate that we can enumerate its elements. All non-zero rational numbers are of the form $\pm\frac{p}{q}$ where $p, q \in \mathbb{N}_+$. These can be organized in a table as follows:



The first row contains all rationals with $q = 1$, the second row contains all rationals with $q = 2$, and so on. At the same time, the first column contains all rationals with $p = 1$, the second column contains all rationals with $p = 2$, and so on. Thus every non-zero rational appears within this table. In fact, every non-zero rational appears infinitely many times in this table, because for every $\pm\frac{p}{q}$, the table also contains $\pm\frac{2p}{2q}$, $\pm\frac{3p}{3q}$, and so on.

The arrows in the table demonstrate a strategy for enumerating the rationals. Since 0 doesn't appear in the table, we start by counting it: we assign to it the number 1. Then we enumerate the elements $\pm\frac{1}{1}$ to which the numbers 2 and 3 are assigned. Next we move on to $\pm\frac{2}{1}$ to which the numbers 4 and 5 are assigned. Then $\pm\frac{1}{2}$, to which 6 and 7 are assigned, $\pm\frac{1}{3}$ to which 8 and 9 are assigned, $\pm\frac{2}{2}$ to which 10 and 11 are assigned, and so on. Eventually to every rational will be assigned a natural number, completing the proof. □

Theorem 1.9: The set $\mathbb{R}$ of real numbers if *not* countable.

Proof. The following proof, *by contradiction*, due to Georg Cantor, goes back to the late 19th century. It's enough to just look at the real numbers between 0 and 1. Each $x \in (0, 1)$ has a decimal representation:

$$x = 0.\, a_1\, a_2\, a_3\, a_4\, a_5\, \dots\dots \qquad \text{where } a_i \in \{0, 1, \dots, 9\} \text{ for every } i \in \mathbb{N}_+.$$

Suppose, by contradiction, that the real numbers in $(0, 1)$ were countable. Then we can enumerate all $x \in (0, 1)$. Let's enumerate them as $r_1, r_2, \dots, r_n, \dots$. Each of these has a decimal representation: $r_n = 0\,.\,c_{n,1}\,c_{n,2}\,c_{n,3}\,c_{n,4}\,\dots\dots$ Let's write the following table, with r_1 on the first line, r_2 on the second, and so on:

$$
\begin{array}{rcccccccc}
r_1 = & 0\,. & \boxed{c_{1,1}} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} & \cdots \\
r_2 = & 0\,. & c_{2,1} & \boxed{c_{2,2}} & c_{2,3} & c_{2,4} & c_{2,5} & \cdots \\
r_3 = & 0\,. & c_{3,1} & c_{3,2} & \boxed{c_{3,3}} & c_{3,4} & c_{3,5} & \cdots \\
r_4 = & 0\,. & c_{4,1} & c_{4,2} & c_{4,3} & \boxed{c_{4,4}} & c_{4,5} & \cdots \\
r_5 = & 0\,. & c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & \boxed{c_{5,5}} & \cdots \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Remember that each digit $c_{i,j}$ is an integer between 0 and 9. Let's now look at the numbers $c_{i,i}$ on the *diagonal*. Consider a real number between 0 and 1 defined as:

$$r = 0.\,d_1\,d_2\,d_3\,d_4\,d_5\,\ldots\ldots \qquad \text{where } d_i \neq c_{i,i} \quad \forall i \in \mathbb{N}_+.$$

Then

$$
\begin{array}{ll}
r \neq r_1 & \text{since the } \textit{first} \text{ digit in their expansions is different} \\
r \neq r_2 & \text{since the } \textit{second} \text{ digit in their expansions is different} \\
r \neq r_3 & \text{since the } \textit{third} \text{ digit in their expansions is different} \\
\vdots & \\
r \neq r_n & \text{since the } n\text{th digit in their expansions is different} \\
\vdots &
\end{array}
$$

Hence r isn't equal to any of the reals that have been enumerated. However, that is a contradiction to the assumption that all reals in $(0, 1)$ have been enumerated. □

This method of proof is called a **diagonal argument**, and has become an important technique for proving various results in the years since Cantor introduced it.

1.4 Cartesian product

Ordered pairs and Cartesian product

Let X and Y be two nonempty sets. Then we define an **ordered pair** to be

$$(x, y)$$

where $x \in X$ and $y \in Y$. The set of all ordered pairs from X and Y is called the **Cartesian product of** X **and** Y and is denoted $X \times Y$:

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

When $X = Y$ we often write X^2 rather than $X \times X$.

The order of elements in an ordered pair is important: the first *component* belongs to X and the second *component* belongs to Y. Thus, the ordered pair (x, y) is *fundamentally different* from the set $\{x, y\}$.

Example 1.6: 1. One of the most important examples of a Cartesian product, which we will often use, is that of the **plane** $\mathbb{R}^2$. This is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

This is called the **Cartesian plane**.

2. A practical example is that of latitude and longitude: we can describe points on Earth in terms of their angle north/south of the equator, and east/west relative to a chosen Prime Meridian (a big circle connecting the North and South poles). Normally this is chosen to be the Greenwich Meridian, passing through the Royal Observatory in Greenwich, England (this choice dates back to 1851). In this system, the Colosseum, for instance, is located at (41.8902°N, 12.4922°E).

An ordered *pair* need not be limited to two components. One can have n elements from n sets, where $n \in \mathbb{N}$:

$$(x_1, x_2, \dots, x_n), \qquad x_i \in X_i,\ i = 1, \dots, n.$$

This is called an n**-tuple**. Then the Cartesian product involves the sets $X_1, \dots, X_n$:

$$X_1 \times X_2 \times \cdots \times X_n.$$

If $X_i = X$ for all $i = 1, \dots, n$, then we simply write $X \times X \times \cdots \times X = X^n$.

1.5 Relations in the Cartesian plane

In the Cartesian plane (we will simply call it *the plane*) $\mathbb{R}^2$ we typically denote the first coordinate by x and the second by y. We can describe subsets of $\mathbb{R}^2$ by equations and inequalities involving x and y. This is best understood using some examples:

Example 1.7: Sets described by equations:

1. The equation $y = 0$ describes the x-axis.
2. The equation $x = 0$ describes the y-axis.
3. The equation $x = y$ describes the line through the origin with slope 1.
4. The equation $x^2 + y^2 = 1$ describes the circle of radius 1 around the origin.

Example 1.8: Sets described by inequalities:

1. The inequality $y < 0$ describes the *lower half plane* excluding the x axis.
2. The inequality $x \geq 0$ describes the *right half plane* including the y axis.
3. The inequality $x < y$ describes the half plane to the left of the line through the origin with slope 1.
4. The inequality $x^2 + y^2 \leq 1$ describes the interior of the circle of radius 1 around the origin, including its boundary.