Absolute value. An important operation that we will frequently encounter is the **absolute value** of a real number x:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

That is, $|x|$ measures the distance of x from 0, regardless of the sign of x. Consequently, we can define the **distance** between $x, y \in \mathbb{R}$ as:

$$|x - y| = \begin{cases} x - y & \text{if } x - y \geq 0 \text{ (i.e., if } x \geq y\text{)}, \\ -(x - y) = y - x & \text{if } x - y < 0 \text{ (i.e., if } x < y\text{)}. \end{cases}$$

1.3.2 Bounded sets

The notion of a bounded set is an extension of the notion of an interval:

Bounded sets

- A subset $A \subseteq \mathbb{R}$ is said to be **bounded from above** if $\exists b \in \mathbb{R}$ such that

  $$x \leq b, \qquad \text{for all } x \in A.$$

  Such a real number b is called an **upper bound** for A.

- $A \subseteq \mathbb{R}$ is said to be **bounded from below** if $\exists a \in \mathbb{R}$ such that

  $$x \geq a, \qquad \text{for all } x \in A.$$

  Such a real number a is called a **lower bound** for A.

- A is called **bounded** if it is bounded from below and from above.

- If A is not bounded from above and not bounded from below, we say that A is **unbounded**.

Lower and upper bounds are not unique

It is very important to observe that both lower and upper bounds are not unique: if a is a lower bound for a subset $A \subseteq \mathbb{R}$, then any $y \leq a$ is also a lower bound. Similarly, if b is an upper bound for A, then any $z \geq b$ is also an upper bound.

Example 1.4:

1. The subset $A_1 = \{-2, 0.5, 7\}$ is bounded. Any number $a \leq -2$ is a lower bound and any number $b \geq 7$ is an upper bound.

2. The subset $A_2 = \mathbb{N}$ is bounded from below but not from above. Any number $a \leq 0$ is a lower bound.

3. The subset $A_3 = \mathbb{Z}$ is unbounded.

4. The subset $A_4 = \{\frac{1}{n} \mid n \in \mathbb{N}_+\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is bounded. Why? We see that a possible upper bound is 1, which belongs to A_4. What about a lower bound? The number 0 is a lower bound, however 0 does not belong to A_4. Below we will see that there is no lower bound that belongs to A_4.

5. The subset $A_5 = \{x \mid |x| > 100\}$ is unbounded.

Let us take a closer look at the lower bound of $A_4 = \{\frac{1}{n} \mid n \in \mathbb{N}_+\}$:

Lemma 1.5: Any lower bound of the set $A_4 = \{\frac{1}{n} \mid n \in \mathbb{N}_+\}$ does not belong to A_4.

Proof. By contradiction, suppose that $\exists a \in A_4$ such that $a \leq x$ for all $x \in A_4$. Since $a \in A_4$, there exists $N \in \mathbb{N}$ such that $a = \frac{1}{N}$. Observe that $\frac{1}{N+1}$ is also an element of A_4 and $\frac{1}{N+1} < \frac{1}{N} = a$. But this contradicts our assumption that a is a lower bound for A_4. Hence no element in A_4 can be a lower bound of A_4. □

In fact, we can prove a stronger statement:

Lemma 1.6: The set $A_4 = \{\frac{1}{n} \mid n \in \mathbb{N}_+\}$ does not have a positive lower bound.

Proof. By contradiction, assume that there exists $r > 0$ that is a lower bound for A_4. Define $N = \lceil \frac{1}{r} \rceil$ to be the first integer greater than or equal to $\frac{1}{r}$. Then $N + 1 > \frac{1}{r}$ and consequently $\frac{1}{N+1} < r$. But $\frac{1}{N+1} \in A_4$, in contradiction to the assumption that r is a lower bound. Hence there is no positive lower bound, and 0 is the *greatest lower bound*. □

Supremum and infimum

Let $A \subset \mathbb{R}$ be a subset.

- The **supremum** (if exists) of A (also called the **least upper bound, l.u.b.**) is the smallest of all upper bounds of A. It is denoted

$$s = \sup A$$

  and it fulfils the following two conditions:

  1. $\forall x \in A,\ x \leq s$,
  2. $\forall r < s\ \exists x \in A$ s.t. $x > r$.

  If there is no such number, we define $\sup A = +\infty$.

- The **infimum** (if exists) of A (also called the **greatest lower bound, g.l.b.**) is the largest of all lower bounds of A. It is denoted

$$\ell = \inf A$$

  and it fulfils the following two conditions:

  1. $\forall x \in A,\ x \geq \ell$,
  2. $\forall r > \ell\ \exists x \in A$ s.t. $x < r$.

  If there is no such number, we define $\inf A = -\infty$.

Note that in the above definition, we defined *the* supremum and *the* infimum, implying that these two numbers are unique. *A priori*, that is not obvious (even though it is true). It requires a proof.

Proposition 1.7: For any subset $A \subseteq \mathbb{R}$, there are unique elements $\ell, s \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ such that $\ell = \inf A$ and $s = \sup A$.

Proof. Exercise. *Hint: prove it by contradiction.* □

The supremum and the infimum might not belong to the set!

It is very important to remember that for a subset $A \subseteq \mathbb{R}$, its supremum and its infimum might *not* belong to it. We have seen it with A_4 above: its infimum is 0, yet $0 \notin A_4$. For $A_2 = \mathbb{N}$, the supremum is $+\infty$, which isn't a number, and in particular isn't an element of A_2.

Keeping in mind the preceding comment, in the case that the supremum and/or infimum *do* belong to the set we give them another name:

Maximum and minimum

Let $A \subset \mathbb{R}$ be a subset. If $\sup A \in A$ then we say that the supremum is *attained*, and it is called the **maximum** of A and denoted

$$\max A.$$

Similarly, if $\inf A \in A$ then we say that the infimum is *attained*, and it is called the **minimum** of A and denoted

$$\min A.$$

1.4 Cartesian product

Ordered pairs and Cartesian product

Let X and Y be two nonempty sets. Then we define an **ordered pair** to be

$$(x, y)$$

where $x \in X$ and $y \in Y$. The set of all ordered pairs from X and Y is called the Cartesian product of X and Y and is denoted $X \times Y$:

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

When $X = Y$ we often write X^2 rather than $X \times X$.