Proof by induction

To prove that a statement $p(n)$ is true for all $n \geq N$ by induction, we need to demonstrate two things:

1. that $p(N)$ (the **base case**) is true,
2. that $p(n)$ being true (the **induction assumption**) implies $p(n+1)$ being true for all $n \geq N$.

1.3 Sets of numbers

We have already seen the definitions of the set $\mathbb{N}$ of natural numbers and the set $\mathbb{Z}$ of integers. We further define

$$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$$

to be the set of positive integers.

The set $\mathbb{Q}$ of rational numbers

We now have the tools to define the set $\mathbb{Q}$ as:

$$\mathbb{Q} = \left\{ r = \frac{p}{q} \;\middle|\; p \in \mathbb{Z},\ q \in \mathbb{N}_+ \right\}$$

Any rational number r has infinitely many representations. For example

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$$

Therefore, we normally choose the (unique) representative such that p and q have no common divisors.

We can also write fractions in base 10 (which is the standard base for us), as

$$r = \pm(c_k 10^k + c_{k-1}10^{k-1} + \cdots c_1 10 + c_0 + c_{-1}10^{-1} + c_{-2}10^{-2} + \cdots)$$

where all coefficients $c_k, c_{k-1}, \ldots, c_0, c_{-1}, \ldots$ can assume any of the values $0, 1, 2, \ldots, 9$. Here are some examples:

$$\begin{aligned}
\frac{1}{2} &= +(0 \cdot 10^k + \cdots + 0 \cdot 10 + 0 + 5 \cdot 10^{-1} + 0 \cdot 10^{-2} + \cdots) = 0.5, \\
-\frac{29}{2} = -14\frac{1}{2} &= -(0 \cdot 10^k + \cdots + 1 \cdot 10 + 4 + 5 \cdot 10^{-1} + 0 \cdot 10^{-2} + \cdots) = -14.5, \\
\frac{1}{3} &= +(0 \cdot 10^k + \cdots + 0 \cdot 10 + 0 + 3 \cdot 10^{-1} + 3 \cdot 10^{-2} + \cdots) = 0.\overline{33},
\end{aligned}$$

where the overline means that the expression repeats itself infinitely:

$$\overline{a_1 a_2 \cdots a_m} = (a_1 a_2 \cdots a_m)(a_1 a_2 \cdots a_m)(a_1 a_2 \cdots a_m)\cdots$$

Decimal representation of rational numbers

Using long division is can be verified that the decimal representation of any rational number is either finite (as in the example for $\frac{1}{2}$) or infinitely repeating (as in the example for $\frac{1}{3}$). The converse is also true: every number that has a finite or repeating representation is rational.

The set $\mathbb{R}$ of real numbers

The world around us is made up of three space dimensions and one time dimension which are all continuous. We move through space and time in a continuous fashion. So, a natural question is whether the rational numbers are enough to describe our world? For instance, as time elapses from 1pm to 2pm, can all intermediate moments be described by rational numbers?

The answer turns out to be *no*. Between 1 and 2, for instance, there exists a number which is the solution of the equation $x^2 = 2$ (we call this number *the square root of* 2 and denote it by $\sqrt{2}$) which is not rational; we normally say that it is **irrational**.

Lemma 1.4 ($\sqrt{2}$ is irrational)**:** The number x satisfying the equation $x^2 = 2$ is irrational.

Proof. We prove this by contradiction. Suppose that there is a rational solution $x = \frac{p}{q}$ (with p and q having no common divisors). Then $\frac{p^2}{q^2} = 2$ so that

$$p^2 = 2q^2.$$

Hence p^2 is even (it is divisible by 2). If p^2 is even then so is p itself. Hence $\exists k \in \mathbb{N}$ such that $p = 2k$ (observe that, in fact, k must be positive, otherwise $x = 0$). We therefore have $(2k)^2 = 2q^2$ which implies that

$$2k^2 = q^2.$$

It follows that q^2 is even, and consequently so is q itself. But this means that p and q have the number 2 as a common divisor, a contradiction. Therefore there is no rational representation for x. □

The real number line. We can write all rational numbers along a single line, placing bigger numbers to the right. Since $1^2 = 1$ and $2^2 = 4$, it seems obvious that $\sqrt{2}$ must lie between 1 and 2. We have thus found an irrational number between 1 and 2,

$$1 < \sqrt{2} < 2.$$

Since it is irrational, it has an infinite representation that *never* repeats. Therefore it is *impossible* to exactly write its value. The beginning of its expansion is:

$$\sqrt{2} = 1.41421356\cdots$$

It follows that

$$\begin{gathered}1.4 < \sqrt{2} < 1.5\\1.41 < \sqrt{2} < 1.42\\1.414 < \sqrt{2} < 1.415\\1.4142 < \sqrt{2} < 1.4143\\\vdots\end{gathered}$$

These are sequences of rational numbers to the left and to the right of $\sqrt{2}$, both of which tend to $\sqrt{2}$, but never reach it. So $\sqrt{2}$ fills a certain gap. Indeed, with the irrational numbers added, we get a continuous line of increasing numbers, called the *real number line*. This property of having a *continuous* line is called *completeness* and we say that the real number line is **complete** (intuitively, it means that there are no gaps).

Actually, there are *many* irrationals. It turns out that there are infinitely many rationals and infinitely many irrationals. They are all intertwined:

- between any two rationals $r_1 < r_2$ there are infinitely many irrationals,
- between any two irrationals $y_1 < y_2$ there are infinitely many rationals.

However, there are *more* irrationals than there are rationals. If we write the set of real numbers (the real number line) as the (disjoint) union of the rationals and the irrationals

$$\mathbb{R} = \mathbb{Q} \sqcup (\mathbb{R} \setminus \mathbb{Q})$$

then both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are infinite, but $\mathbb{R} \setminus \mathbb{Q}$ is 'bigger' in a certain sense. This statement is made rigorous using a mathematical theory called *measure theory*. It can be understood intuitively as follows: if a number is chosen randomly between, say, 0 and 1, then it is *almost surely* an irrational number (in other words, the probability of choosing a rational number is zero). So within the real number line there are 'more' irrationals. However there are also rationals *everywhere*. We say that **the rational numbers $\mathbb{Q}$ are dense within the reals** $\mathbb{R}$.

1.3.1 The ordering of real numbers

The real numbers are a **totally ordered set**: $\forall x, y \in \mathbb{R}$, one (and *only* one) of the following properties holds:

$$x = y \quad \text{or} \quad x < y \quad \text{or} \quad x > y.$$

We often use the following symbols for these important subsets of $\mathbb{R}$:

$$\begin{aligned}\mathbb{R}_+ &= \{x \in \mathbb{R} \mid x > 0\}\\\mathbb{R}_- &= \{x \in \mathbb{R} \mid x < 0\}\\\mathbb{R}_* &= \{x \in \mathbb{R} \mid x \geq 0\} = \mathbb{R}_+ \cup \{0\}\end{aligned}$$

Infinity. It is convenient to introduce a symbol to help us express the fact that there is no greatest number. We thus introduce the symbol for **plus infinity**

$$+\infty,$$

which is thought of as an object that is greater than any real number. Similarly, **minus infinity**

$$-\infty$$

symbolizes an object that is smaller than any real number. Note that these are *not* numbers.

Intervals. The notion of an *interval* – a part of the real number line – will be very important:

Intervals

Let $a, b \in \mathbb{R}$ with $a \leq b$. Then the **closed interval** between a and b is defined as

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Let $a, b \in \mathbb{R}$ with $a < b$. Then the **open interval** between a and b is defined as

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

That is, closed intervals include all points between a and b, including a and b themselves. Open intervals do not include the endpoints. The points between a and b (excluding a and b themselves) are called **interior points**.

We can also define intervals that are closed on one end and open on the other. Let $a < b$, then

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

Other important sets are **half-lines**, which are sets that have a lower/upper limit only on one side. Here the symbols for plus or minus infinity come in handy:

$$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\},$$
$$(a, +\infty) = \{x \in \mathbb{R} \mid a < x\},$$
$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\},$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}.$$

The entire real line is often represented as the set of all points that are greater than $-\infty$ and less than $+\infty$:

$$\mathbb{R} = (-\infty, +\infty).$$