Chapter 1

Basic Notions

1.1 Sets

A **set** is a collection of elements. We denote sets with curly brackets $\{\cdots\}$, with the elements listed within the brackets. As an example, consider the set of students in the class *Mathematical Analysis 1*. This set can either be written explicitly,

$$X = \{\text{Bob}, \text{Lucy}, \text{Andrew}, \text{Giulia}\}$$

or it can be defined using a rule:

$$X = \{\text{all people who are students of } \textit{Mathematical Analysis 1}\}.$$

Some important sets of numbers that we will often encounter are

$$\begin{aligned}
\mathbb{N} &= \text{set of natural numbers} = \{0, 1, 2, \dots\}\\
\mathbb{Z} &= \text{set of integer numbers} = \{0, \pm 1, \pm 2, \dots\}\\
\mathbb{Q} &= \text{set of rational numbers} = \{\text{can you think how to define?}\}\\
\mathbb{R} &= \text{set of real numbers} = \{\text{can you think how to define?}\}\\
\mathbb{C} &= \text{set of complex numbers} = \{\text{can you think how to define?}\}
\end{aligned}$$

Another important set is the **empty set** which contains no elements. It is denoted $\emptyset$.

Basic notation

- **Element of**: if x is an element of X we write $x \in X$
- **Not element of**: if x is *not* an element of X we write $x \notin X$
- **Subset**: if A is a subset of X (i.e. any element of A is also an element of X) we write $A \subseteq X$ or $X \supseteq A$

  In this case it is possible that $A = X$.
- **Proper subset**: if A is a *proper* subset of X we write $A \subset X$ or $X \supset A$

  In this case $A \neq X$ (i.e. there exists $x \in X$ and $x \notin A$).

Lemma 1.1: For some set X and subsets $A, B \subseteq X$, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Proof. By contradiction, assume that $A \neq B$. Then, without loss of generality, there exists $x \in X$ such that $x \in A$ and $x \notin B$. But then it is not true that $A \subseteq B$. The contradiction assumption must therefore be false, i.e. $A = B$. □

Characteristic property

The elements of a subset $A \subseteq X$ can often be characterized by a mathematical property that they satisfy. This property is denoted $p(x)$, and we write

$$A = \{x \in X \mid p(x)\}.$$

For example, if $p(x) =$ 'x is even', then

$$A = \{x \in \mathbb{N} \mid x \text{ is even}\} = \{0, 2, 4, \dots\} \subseteq \mathbb{N}.$$

Operations on sets

- **Complement**: if $A \subseteq X$ then we define its *complement* to be

$$A^C = \complement A = \{x \in X \mid x \notin A\}.$$

- **Union**: for two sets $A \subseteq X$ and $B \subseteq X$ we define their *union* to be

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$$

- **Intersection**: for two sets $A \subseteq X$ and $B \subseteq X$ we define their *intersection* to be

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$$

- **Difference**: for two sets $A \subseteq X$ and $B \subseteq X$ we define their *difference* to be

$$A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}$$

- **Symmetric Difference**: for two sets $A \subseteq X$ and $B \subseteq X$ we define their *symmetric difference* to be

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

- **Disjoint Union**: for two sets $A \subseteq X$ and $B \subseteq X$ whose intersection is empty, we often replace the symbol $\cup$ by

$$A \sqcup B \quad \text{or} \quad A \dot{\cup} B$$

Lemma 1.2 (Properties of $\cap$ and $\cup$)**:** For some set X and subsets $A, B, C \subseteq X$ the operations $\cap$ and $\cup$ satisfy: