MATHEMATICAL ANALYSIS 1
HOMEWORK 6

(1) In this problem we prove the **Squeeze Theorem** for a finite point x_0, and three functions f, g, h satisfying $f \le g \le h$ near x_0, with f and h having the limit ℓ as $x \to x_0$ (you are guided in the steps below):
  (a) State the theorem (and state that you shall prove it only in the case $x_0 \in \mathbb{R}$).
  (b) Fix $\varepsilon > 0$.
  (c) With this ε write the definition of what it means that $\lim_{x\to x_0} f(x) = \ell$.
  (d) Write the definition of what it means that $\lim_{x\to x_0} h(x) = \ell$.
  (e) Using the previous two steps, find a neighborhood of x_0 (depending on ε) for which you can write a condition for convergence to ℓ for the function g.
  (f) Conclude that, since $\varepsilon > 0$ was arbitrary, the theorem follows by the definition of the limit (applied to g).

(2) Prove that
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$
Hint: multiply both the numerator and denominator by $1 + \cos x$.

(3) Compute the following limits:
  (a) $\lim_{x\to+\infty} \frac{\cos x}{\sqrt{x}}$
  (b) $\lim_{x\to+\infty} \frac{\lfloor x \rfloor}{x}$
  (c) $\lim_{x\to 0} \frac{x-\tan x}{x^2}$
  (d) $\lim_{x\to e} \frac{\ln x - 1}{x-e}$ *(hint: take $y = x - e$)*
  (e) $\lim_{x\to+\infty} \frac{x+3}{x^3-2x+5}$
  (f) $\lim_{x\to 0} \frac{\sin^2 x}{x}$
  (g) $\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x}$ *(hint: take $y = 1 - x$)*
  (h) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{\sin x}$
  (i) $\lim_{x\to 0+} \frac{2^{2x}-1}{2x}$
  (j) $\lim_{x\to 1} \frac{\ln x}{e^x - e}$ *(hint: take $y = x - 1$)*
  (k) $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$
  (l) $\lim_{x\to-\infty} x e^{\sin x}$

(4) Determine how the following sequences $\{a_n\}_{n\in\mathbb{N}}$ behave for large n:
  (a) $a_n = n - \sqrt{n}$
  (b) $a_n = \frac{(2n)!}{n!}$
  (c) $a_n = \frac{(2n)!}{(n!)^2}$
  (d) $a_n = \binom{n}{3}\frac{6}{n^3}$
  (e) $a_n = 2^n \sin(2^{-n}\pi)$
  (f) $a_n = n \cos\left(\frac{n+1}{n}\cdot\frac{\pi}{2}\right)$

(5) Use the fact that $\lim_{x\to\pm\infty}(1+\frac{1}{x})^x = e$ to prove that for $a \neq 0$
$$\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

(6) Let $f, g : \mathbb{R} \to \mathbb{R}$. Show that $f \sim g$ as $x \to x_0$ if and only if $f = g + o(g)$ as $x \to x_0$.

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be infinite or infinitesimal at x_0.
  (a) State the definition of the *order* α of f at x_0 with respect to a function $\varphi : \mathbb{R} \to \mathbb{R}$.
  (b) Prove that the order α is unique.

(8) Determine the order and the principal part with respect to $\varphi(x) = \frac{1}{x}$ as $x \to +\infty$ of the function $f(x) = \sin(\sqrt{x^2-1} - x)$.

(9) As $x \to +\infty$, the function $f(x) = \ln(9 + \sin\frac{2}{x}) - 2\ln 3$ can be written as $f(x) = \frac{b}{x^\alpha} + o(x^{-\alpha})$. Find b and α.

(10) Determine the order and the principal part with respect to $\varphi(x) = x$ as $x \to 0$ of the function $f(x) = \frac{e^x}{1+x^2} - 1$.

(1) (a) **Squeeze Theorem:** Let f, g, h be functions defined on a neighborhood of $x_0 \in \mathbb{R}$ (possibly excluding x_0 itself) such that $f(x) \le g(x) \le h(x)$ for all x in this neighborhood. If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \ell$, then $\lim_{x\to x_0} g(x) = \ell$.
(b) Let $\varepsilon > 0$ be given.
(c) Since $\lim_{x\to x_0} f(x) = \ell$, there exists $\delta_1 > 0$ such that for all x with $0 < |x - x_0| < \delta_1$, we have $|f(x) - \ell| < \varepsilon$, which implies $\ell - \varepsilon < f(x) < \ell + \varepsilon$.
(d) Since $\lim_{x\to x_0} h(x) = \ell$, there exists $\delta_2 > 0$ such that for all x with $0 < |x - x_0| < \delta_2$, we have $|h(x) - \ell| < \varepsilon$, which implies $\ell - \varepsilon < h(x) < \ell + \varepsilon$.
(e) Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all x with $0 < |x - x_0| < \delta$, we have:

$$\ell - \varepsilon < f(x) \le g(x) \le h(x) < \ell + \varepsilon$$

Thus $|g(x) - \ell| < \varepsilon$.
(f) Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{x\to x_0} g(x) = \ell$.

(2) *Proof.* Multiply numerator and denominator by $1 + \cos x$:

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}$$

As $x \to 0$, we have:
$\frac{\sin x}{x} \to 1$
$\frac{1}{1+\cos x} \to \frac{1}{2}$
Therefore:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

□

(3) (a) $\lim_{x\to+\infty} \frac{\cos x}{\sqrt{x}} = 0$ (bounded numerator, denominator $\to +\infty$)
(b) $\lim_{x\to+\infty} \frac{\lfloor x \rfloor}{x} = 1$ (using the Squeeze Theorem, since $x - 1 < \lfloor x \rfloor \le x$)
(c) $\lim_{x\to 0} \frac{x - \tan x}{x^2}$.
The function $f(x) = \frac{x-\tan x}{x^2}$ is odd, so we evaluate $\lim_{x\to 0^+} f(x)$. For $0 < x < \frac{\pi}{2}$, the inequality $\sin x < x < \tan x$ holds. Subtracting $\tan x$ gives $\sin x - \tan x < x - \tan x < 0$. Dividing by $x^2 > 0$:

$$\frac{\sin x - \tan x}{x^2} < \frac{x - \tan x}{x^2} < 0.$$

We evaluate the limit of the lower bound:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sin x - \tan x}{x^2} &= \lim_{x\to 0^+} \frac{\sin x - \frac{\sin x}{\cos x}}{x^2} \\
&= \lim_{x\to 0^+} \frac{\sin x(\cos x - 1)}{x^2 \cos x} \\
&= \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right) \cdot \left(\frac{1}{\cos x}\right) \cdot \left(-x \cdot \frac{1-\cos x}{x^2}\right) \\
&= 1 \cdot \frac{1}{1} \cdot \left(-0 \cdot \frac{1}{2}\right) = 0.
\end{aligned}$$

By the **Squeeze Theorem**, since the function is bounded between 0 and a function tending to 0, we conclude that:

$$\lim_{x\to 0^+} \frac{x - \tan x}{x^2} = 0.$$

Thus, the required limit is 0.
(d) Let $y = x - e$, then $x = y + e$:

$$\lim_{x\to e} \frac{\ln x - 1}{x - e} = \lim_{y\to 0} \frac{\ln(y+e) - 1}{y} = \lim_{y\to 0} \frac{\ln e + \ln(y/e + 1) - 1}{y} = \lim_{y\to 0} \frac{\ln(y/e+1)}{y} = \frac{1}{e}$$

(e)

$$\begin{aligned}
\lim_{x\to+\infty}\frac{x+3}{x^3-2x+5} &= \lim_{x\to+\infty}\frac{\frac{x}{x^3}+\frac{3}{x^3}}{\frac{x^3}{x^3}-\frac{2x}{x^3}+\frac{5}{x^3}}\\
&= \lim_{x\to+\infty}\frac{\frac{1}{x^2}+\frac{3}{x^3}}{1-\frac{2}{x^2}+\frac{5}{x^3}}\\
&= \frac{\lim_{x\to+\infty}\left(\frac{1}{x^2}+\frac{3}{x^3}\right)}{\lim_{x\to+\infty}\left(1-\frac{2}{x^2}+\frac{5}{x^3}\right)}\\
&= \frac{\lim_{x\to+\infty}\frac{1}{x^2}+\lim_{x\to+\infty}\frac{3}{x^3}}{\lim_{x\to+\infty}1-\lim_{x\to+\infty}\frac{2}{x^2}+\lim_{x\to+\infty}\frac{5}{x^3}}\\
&= \frac{0+0}{1-0+0}\\
&= \frac{0}{1}\\
&= 0
\end{aligned}$$

(f) $\lim_{x\to0}\frac{\sin^2 x}{x}=\lim_{x\to0}\frac{\sin x}{x}\cdot\sin x=1\cdot0=0$

(g) Let $y=1-x$, then $x=1-y$:

$$\lim_{x\to1}\frac{\cos(\frac{\pi}{2}x)}{1-x}=\lim_{y\to0}\frac{\cos(\frac{\pi}{2}(1-y))}{y}=\lim_{y\to0}\frac{\cos(\frac{\pi}{2}-\frac{\pi}{2}y)}{y}=\lim_{y\to0}\frac{\sin(\frac{\pi}{2}y)}{y}=\frac{\pi}{2}$$

(h) Multiply numerator and denominator by the conjugate:

$$\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{\sin x}=\lim_{x\to0}\frac{(1+\tan x)-(1-\tan x)}{\sin x(\sqrt{1+\tan x}+\sqrt{1-\tan x})}$$

$$=\lim_{x\to0}\frac{2\tan x}{\sin x(\sqrt{1+\tan x}+\sqrt{1-\tan x})}=\lim_{x\to0}\frac{2}{\cos x(\sqrt{1+\tan x}+\sqrt{1-\tan x})}=\frac{2}{1\cdot(1+1)}=1$$

(i) $\lim_{x\to0^+}\frac{2^{2x}-1}{2x}=\lim_{y\to0^+}\frac{2^y-1}{y}=\ln 2$

(j) Let $y=x-1$, then $x=y+1$:

$$\lim_{x\to1}\frac{\ln x}{e^x-e}=\lim_{y\to0}\frac{\ln(1+y)}{e^{y+1}-e}=\lim_{y\to0}\frac{\ln(1+y)}{e(e^y-1)}=\frac{1}{e}\lim_{y\to0}\frac{\ln(1+y)}{y}\frac{y}{e^y-1}=\frac{1}{e}$$

(k) $\lim_{x\to0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to0}\frac{e^{-x}(e^{2x}-1)}{\sin x}=\lim_{x\to0}e^{-x}\frac{e^{2x}-1}{2x}\cdot2\cdot\frac{x}{\sin x}=2$

(l) $xe^{\sin x}\le xe^{-1}=x/e$ for all $x<-1$, and $\lim_{x\to-\infty}x/e=(1/e)\lim_{x\to-\infty}x=-\infty$ hence $\lim_{x\to-\infty}xe^{\sin x}=-\infty$.

(4) (a) $a_n=n-\sqrt{n}\to+\infty$

$$\begin{aligned}
\lim_{n\to\infty}a_n &= \lim_{n\to\infty}(n-\sqrt{n})\\
&= \lim_{n\to\infty}n\left(1-\frac{1}{\sqrt{n}}\right)\\
&= \lim_{n\to\infty}n\,(1-0)=+\infty
\end{aligned}$$

(b) $a_n=\frac{(2n)!}{n!}\to+\infty$ very rapidly (Ratio Test)

$$\begin{aligned}
L &= \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\\
&= \lim_{n\to\infty}\frac{(2n+2)!}{(n+1)!}\cdot\frac{n!}{(2n)!}\\
&= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{n+1}\\
&= \lim_{n\to\infty}2(2n+1)=+\infty
\end{aligned}$$

Since $L = +\infty > 1$, the sequence diverges to $+\infty$.

(c) $a_n = \frac{(2n)!}{(n!)^2} \to +\infty$ (Ratio Test)

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \\ &= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} \\ &= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = 4 \end{aligned}$$

Since $L = 4 > 1$, the sequence diverges to $+\infty$.

(d) $a_n = \binom{n}{3}\frac{6}{n^3} \to 1$

$$\begin{aligned} a_n &= \binom{n}{3}\frac{6}{n^3} \\ &= \frac{n(n-1)(n-2)}{6} \cdot \frac{6}{n^3} \\ &= \frac{n^3}{n^3} - \frac{3n^2}{n^3} + \frac{2n}{n^3} \\ &= 1 - \frac{3}{n} + \frac{2}{n^2} \\ \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(1 - \frac{3}{n} + \frac{2}{n^2}\right) \\ &= 1 - 0 + 0 \\ &= 1 \end{aligned}$$

(e) $a_n = 2^n \sin(2^{-n}\pi) \to \pi$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2^n \sin(2^{-n}\pi) \\ &= \lim_{n\to\infty} \left(\frac{\sin(2^{-n}\pi)}{2^{-n}\pi}\right) \cdot \left(2^n \cdot 2^{-n}\pi\right) \\ &= \lim_{n\to\infty} \left(\frac{\sin(2^{-n}\pi)}{2^{-n}\pi}\right) \cdot (\pi) \\ &= (1)\cdot\pi = \pi \qquad \left(\text{using } \lim_{u\to 0} \frac{\sin u}{u} = 1\right) \end{aligned}$$

(f) $a_n = n\cos\left(\frac{n+1}{n}\cdot\frac{\pi}{2}\right) \to -\frac{\pi}{2}$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} n\cos\left(\frac{\pi}{2} + \frac{\pi}{2n}\right) \\ &= \lim_{n\to\infty} -n\sin\left(\frac{\pi}{2n}\right) \qquad \left(\text{using } \cos(\frac{\pi}{2}+\theta) = -\sin\theta\right) \\ &= \lim_{n\to\infty} -\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right) \cdot \left(n\cdot\frac{\pi}{2n}\right) \\ &= \lim_{n\to\infty} -\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right) \cdot \left(\frac{\pi}{2}\right) \\ &= -(1)\cdot\frac{\pi}{2} = -\frac{\pi}{2} \end{aligned}$$

(5) *Proof.* Let $y = \frac{x}{a}$. Then as $x \to \pm\infty$, $y \to \pm\infty$ (since $a \neq 0$). Then:

$$\left(1 + \frac{a}{x}\right)^x = \left(1 + \frac{1}{y}\right)^{ay} = \left[\left(1 + \frac{1}{y}\right)^y\right]^a$$

Since $\lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y = e$, we have:

$$\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

□

(6) *Proof.* ($\Rightarrow$) If $f \sim g$ as $x \to x_0$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$. Let $h(x) = f(x) - g(x)$. Then:

$$\frac{h(x)}{g(x)} = \frac{f(x)}{g(x)} - 1 \to 1 - 1 = 0$$

So $h(x) = o(g(x))$, which means $f(x) = g(x) + o(g(x))$.

($\Leftarrow$) If $f(x) = g(x) + o(g(x))$, then $\frac{f(x)}{g(x)} = 1 + \frac{o(g(x))}{g(x)} \to 1 + 0 = 1$, so $f \sim g$. □

(7) (a) The *order* α of f at x_0 with respect to φ is defined as the number $\alpha \in \mathbb{R}$ such that:

$$\lim_{x\to x_0} \frac{f(x)}{\varphi(x)^\alpha} = L \neq 0$$

where L is a finite nonzero constant.

(b) *Proof.* Suppose there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$ such that:

$$\lim_{x\to x_0} \frac{f(x)}{\varphi(x)^\alpha} = L \neq 0 \quad \text{and} \quad \lim_{x\to x_0} \frac{f(x)}{\varphi(x)^\beta} = M \neq 0$$

Then:

$$\frac{f(x)}{\varphi(x)^\alpha} = \frac{f(x)}{\varphi(x)^\beta} \cdot \varphi(x)^{\beta-\alpha}$$

Taking limits:

$$L = M \cdot \lim_{x\to x_0} \varphi(x)^{\beta-\alpha}$$

If $\beta > \alpha$, then $\varphi(x)^{\beta-\alpha} \to 0$ or ∞ depending on whether f is infinitesimal or infinite, so $L = 0$ or ∞, contradicting $L \neq 0$. Similarly if $\alpha > \beta$. Therefore, $\alpha = \beta$. □

(8) Note first of all that

$$\lim_{x\to+\infty} (\sqrt{x^2-1} - x) = \lim_{x\to+\infty} \frac{x^2 - 1 - x^2}{\sqrt{x^2-1} + x} = \lim_{x\to+\infty} \frac{-1}{\sqrt{x^2-1} + x} = 0,$$

hence the function $f(x)$ is infinitesimal as $x \to +\infty$. In addition,

$$\lim_{x\to+\infty} \frac{\sin(\sqrt{x^2-1} - x)}{\sqrt{x^2-1} - x} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

Then

$$\lim_{x\to+\infty} x^\alpha \sin(\sqrt{x^2-1} - x) = \lim_{x\to+\infty} x^\alpha (\sqrt{x^2-1} - x) \frac{\sin(\sqrt{x^2-1} - x)}{\sqrt{x^2-1} - x} = \lim_{x\to+\infty} x^\alpha (\sqrt{x^2-1} - x).$$

Computing the right-hand-side limit gives

$$\begin{aligned}\lim_{x\to+\infty} x^\alpha (\sqrt{x^2-1} - x) &= \lim_{x\to+\infty} x^\alpha \frac{-1}{\sqrt{x^2-1} + x} \\ &= \lim_{x\to+\infty} \frac{-x^\alpha}{x\left(\sqrt{1 - \frac{1}{x^2}} + 1\right)}\end{aligned}$$

Choosing $\alpha = 1$, the order is 1 and the principal part is $p(x) = -\frac{1}{2x}$.

(9) $\ln(9+\sin\frac{2}{x}) - 2\ln 3 = \ln 9(1+\frac{1}{9}\sin\frac{2}{x}) - \ln 9 = \ln(1+\frac{1}{9}\sin\frac{2}{x})$.
For $x \to +\infty$, $\frac{1}{9}\sin\frac{2}{x} \sim \frac{2}{9x}$.
For $y \to 0$, $\ln(1+y) \sim y$.
Hence

$$\lim_{x\to+\infty} x^\alpha f(x) = \lim_{x\to+\infty} x^\alpha \frac{1}{9}\sin\frac{2}{x} = \lim_{x\to+\infty} \frac{2x^\alpha}{9x} = \frac{2}{9}$$

if $\alpha = 1$. So $\alpha = 1$ and $b = \frac{2}{9}$.

(10) Using the relation $e^x = 1 + x + o(x)$ for $x \to 0$ we have

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{x^\alpha} &= \lim_{x\to 0}\frac{e^x - 1 - x^2}{x^\alpha(1+x^2)}\\ &= \lim_{x\to 0}\frac{e^x - 1 - x^2}{x^\alpha}\\ &= \lim_{x\to 0}\left(\frac{e^x - 1}{x^\alpha} - x^{2-\alpha}\right) = 1\end{aligned}$$

for $\alpha = 1$. The order of f is 1 and the principal part is $p(x) = x$.