MATHEMATICAL ANALYSIS 1
HOMEWORK 1

(1) Prove that there are infinitely many prime numbers (Euclid's Theorem).
(2) Prove that the solution of the equation $x^2 = 7$ is irrational.
(3) Prove that for any set X and for any two subsets $A, B \subset X$, we have $(A \cup B)^C = A^C \cap B^C$.
(4) Solve the inequality: $\sqrt{|x^2 - 4|} - x \geq 0$.
(5) Let $P \subseteq \mathbb{R}$ be the set of prime numbers.
   (a) Is P bounded from above? from below? what is its infimum? supremum? do either the supremum or infimum belong to P?
   (b) Let $Q = \{x \in \mathbb{R} \mid x^{-1} \in P\}$. Is Q bounded from above? from below? what is its infimum? supremum? do either the supremum or infimum belong to Q?
(6) Consider the following subset of $\mathbb{R}$:

$$A = \left\{ x \in \mathbb{R} \quad \middle| \quad 0 \leq x < 1 \quad \text{or} \quad x = \frac{2n-3}{n-1},\ n \in \mathbb{N} \setminus \{0, 1\} \right\}.$$

Is A bounded from above? from below? what is its infimum? supremum? do either the supremum or infimum belong to A?

Solutions

(1) We prove by contradiction. Suppose there are finitely many primes $p_1, p_2, \ldots, p_n$. Let $N = p_1 p_2 \cdots p_n + 1$. Then $N > 1$ and N is not divisible by any p_i (since division by p_i leaves remainder 1). Thus either N is prime or has a prime divisor different from all p_i, contradicting the assumption that $p_1, \ldots, p_n$ are all primes.

(2) We prove by contradiction. Suppose $x^2 = 7$ has a rational solution $x = \frac{p}{q}$ where $p, q \in \mathbb{N}$ are have no common divisors. Then $p^2 = 7q^2$, so $7 \mid p^2$. Since 7 is prime, $7 \mid p$, say $p = 7k$. Then $49k^2 = 7q^2$, so $7k^2 = q^2$ and $7 \mid q^2$, hence $7 \mid q$. This contradicts the assumption that p and q have no common divisors.

(3) As in the proof we did in class, to show equality of two sets we show that they both contain one another. Let $x \in (A \cup B)^C$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Thus $x \in A^C$ and $x \in B^C$, so $x \in A^C \cap B^C$. Conversely, if $x \in A^C \cap B^C$, then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$, hence $x \in (A \cup B)^C$.

(4) The inequality is $\sqrt{|x^2 - 4|} \geq x$. For $x \leq 0$, this holds automatically. For $x > 0$, we require $|x^2 - 4| \geq x^2$, which gives $x^2 - 4 \leq -x^2$ or $x^2 - 4 \geq x^2$. The latter is impossible. The former gives $2x^2 \leq 4$, so $x^2 \leq 2$, i.e., $0 < x \leq \sqrt{2}$. Therefore the solution is $x \in (-\infty, \sqrt{2}]$.

(5) (a) P is not bounded from above (there are infinitely many primes). P is bounded from below by 0 (or by 2). We have $\inf P = 2 \in P$, and $\sup P$ does not exist in $\mathbb{R}$ (or $\sup P = +\infty$).
  (b) $Q = \{\frac{1}{p} \mid p \in P\}$ is bounded from above by $\frac{1}{2}$ and from below by 0. We have $\sup Q = \frac{1}{2} \in Q$ and $\inf Q = 0 \notin Q$ (since Q consists of reciprocals of primes, $0 \notin Q$).

(6) Note that $\frac{2n-3}{n-1} = \frac{2(n-1)-1}{n-1} = 2 - \frac{1}{n-1}$. For $n \geq 2$, this sequence increases toward 2 from below. Thus $A = [0,1) \cup \{1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots\}$ where the sequence approaches 2. We have $\inf A = 0 \in A$, and $\sup A = 2 \notin A$ (since all values $\frac{2n-3}{n-1} < 2$ for $n \geq 2$). A is bounded both from above and from below.