MATHEMATICAL ANALYSIS 1
HOMEWORK 5

(1) For the function $f(x) = \sin \frac{1}{x}$, find all $x_n \in (0,1)$ for which $f(x_n) = 1$ and all $y_n \in (0,1)$ for which $f(y_n) = -1$. Conclude that $\lim_{x \to 0^+} f(x)$ does not exist.
(2) Prove that the ceiling function $f(x) = \lceil x \rceil$ is left-continuous at every $x_0 \in \mathbb{R}$.
(3) Prove the following proposition:

Proposition: Let $f : \mathbb{R} \to \mathbb{R}$ be defined in a neighborhood of x_0 (possibly not at x_0 itself). Then

$$\lim_{x \to x_0} f(x) = L \qquad \Leftrightarrow \qquad \lim_{x \to x_0^+} f(x) = L \quad \text{and} \quad \lim_{x \to x_0^-} f(x) = L$$

where L can be any number or $\pm\infty$. Moreover, the function is continuous at x_0 if and only if it is both right- and left-continuous at x_0.

(4) Let $f(x) = 5 + 2x \sin \frac{1}{x}$. Does the limit $\lim_{x \to +\infty} f(x)$ exist? Prove your answer.
(5) Let $f : \mathbb{R} \to \mathbb{R}$. Prove that if $\lim_{x \to +\infty} f(x) > 0$ then there exists $M > 0$ s.t. $f > 0$ on the set $\{x \in \mathbb{R} \,:\, x > M\}$. *Hint: we proved a very similar result in class.*
(6) Consider the sequence $a_n = \arctan\left(\frac{5n+6}{n+1}\right)$, $n \in \mathbb{N}$.
   (a) Is this a monotone sequence?
   (b) Find its infimum and supremum.
   (c) Do the minimum and maximum exist? If so, what are they?
   (d) Does the limit $\lim_{n \to \infty} a_n$ exist? If so, what is it?
(7) Using the definition of the limit prove the following:
   (a) $\lim_{x \to 1} (2x^2 + 3) = 5$
   (b) $\lim_{n \to \infty} \frac{n^2}{1-2n} = -\infty$
(8) Determine the values of $\alpha \in \mathbb{R}$ for which the following functions are continuous on their domains. Explain your answers.
   (a) $f(x) = \begin{cases} \alpha \sin(x + \frac{\pi}{2}) & \text{for } x > 0 \\ 2x^2 + 3 & \text{for } x \leq 0 \end{cases}$
   (b) $f(x) = \begin{cases} 3e^{\alpha x - 1} & \text{for } x \geq 1 \\ x + 2 & \text{for } x < 1 \end{cases}$
(9) Compute the following limits:
   (a) $\lim_{x \to 0} \frac{x^4 - 2x^3 + 5x}{x^5 - x}$
   (b) $\lim_{x \to -1} \frac{x+1}{\sqrt{6x^2+3} + 3x}$
   (c) $\lim_{x \to +\infty} (\sqrt{x+1} - \sqrt{x})$
   (d) $\lim_{x \to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$
(10) Determine the domain and the behvior at the end-points of the domain of the following functions:
   (a) $f(x) = \frac{x^3 - x^2 + 3}{x^2 + 3x + 2}$
   (b) $f(x) = \sqrt[3]{x} e^{-x^2}$ *(hint: you may use the fact that* $\lim_{x \to +\infty} (\frac{1}{3} \ln x - x^2) = -\infty$*)*