MATHEMATICAL ANALYSIS 1
HOMEWORK 4

(1) Prove De Moivre's formula (*without* using the exponential form of complex numbers): for any complex number $z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in \mathbb{R}$, and any $n \in \mathbb{N}$,
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$
Hint: by induction.

(2) Let $z = 2 - 2i$ and $w = -1 + i\sqrt{3}$.
- (a) Write z and w in exponential form.
- (b) Compute $z \cdot w$ and express the result in both Cartesian and exponential forms.
- (c) Compute $\frac{z}{w}$ and express the result in both Cartesian and exponential forms.
- (d) Find the complex conjugate $\overline{z}$ and compute $z \cdot \overline{z}$.

(3) Let $z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
- (a) Write z and w in Cartesian form.
- (b) Compute z^2 and w^3 using both De Moivre's formula and direct multiplication.
- (c) Verify that $|z \cdot w| = |z| \cdot |w|$ and $\arg(z \cdot w) = \arg(z) + \arg(w)$.

(4) Find all complex numbers z that satisfy:
- (a) $z^2 = -4$
- (b) $z^3 = 8i$
- (c) $z^4 = -16$

Express your answers in both Cartesian and exponential forms.

(5) Let $z = 1 + i\sqrt{3}$.
- (a) Write z in exponential form.
- (b) Compute z^4 and express the result in both exponential and Cartesian forms.
- (c) Find all complex solutions to $w^3 = z$.

(6) Let $z = \frac{1+i}{1-i}$.
- (a) Simplify z to Cartesian form.
- (b) Write z in exponential form.
- (c) Compute z^{2023} (hint: use the exponential form).

(7) Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. Prove that it can have at most one limit as $n \to \infty$ (hint: by contradiction).

(8) We proved that a monotone increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ has a limit (which can be finite or infinite) and that it is equal to the supremum of the sequence (see the lecture notes for the precise statement). State and prove the analogous result for a monotone *decreasing* sequence.

(9) Let $\{a_n\}_{n\in\mathbb{N}}$ be a real-valued sequence. Suppose that the limit $\lim_{n\to\infty} |a_n|$ exists. Does $\lim_{n\to\infty} a_n$ exist? If so, prove it. If not, give a counterexample.

(10) Let $x_0 \in \mathbb{R}$.
- (a) Let $\varepsilon > 0$. Define the ε-neighborhood of x_0 (this is another name for the neighborhood of x_0 of size ε).
- (b) Prove that the intersection of two neighborhoods of x_0 is also a neighborhood of x_0.

(11) Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_n = \frac{n^2+(-1)^n n}{n^2+1}$.
- (a) Determine whether the sequence has a finite limit, infinite limit, or is indeterminate.
- (b) Prove your answer using the definition of limit.

(12) Let $f(x) = \frac{2x^2-3x+1}{x^2+4}$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Prove your answers.