MATHEMATICAL ANALYSIS 1
HOMEWORK 3

(1) Prove the following lemma:

Lemma. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be monotonically increasing on some $A \subseteq \mathbb{R}$. Then $f + g$ is also monotonically increasing on A. If either f or g are* strictly *increasing on A, then so is $f + g$. The same statements hold if we replace everywhere the word 'increasing' with the word 'decreasing'.*

(2) Prove the following lemma:

Lemma. *Let $f, g : \mathbb{R} \to \mathbb{R}$. Then:*

$$\begin{aligned} f, g \text{ are both monotone increasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are both monotone decreasing} \quad &\Rightarrow \quad g \circ f \text{ is monotone increasing.} \\ f, g \text{ are monotone of different kinds} \quad &\Rightarrow \quad g \circ f \text{ is monotone decreasing.} \end{aligned}$$

(3) Show that if $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is monotone increasing on A, then $-f$ is monotone decreasing on A.

(4) Let t_c and s_c be the translation and scaling functions, respectively. Let r be the reflection function. Consider the function $f(x) = x^3$ restricted to the interval $[-1, 1]$. Sketch the following:
   (a) $f \circ t_c$ for $c = -1, 0, 1$
   (b) $t_c \circ f$ for $c = -1, 0, 1$
   (c) $f \circ s_c$ for $c = \frac{1}{2}, 2$
   (d) $s_c \circ f$ for $c = \frac{1}{2}, 2$
   (e) The difference $f \circ r - r \circ f$.

(5) Sketch the graphs of the following functions (choose a relevant part of the graph to sketch):
   (a) $\sin(x)$, $\sin(2x)$, $\sin(-x)$
   (b) $\log_{\frac{1}{5}} x$, $\log_5 x$, $\log_5(x^2)$
   (c) $\cos(x)$, $\cos(2x)$, $\cos(x - \frac{\pi}{2})$

(6) For the following pairs of functions f and g, compute $f \circ g$ and $g \circ f$, and determine their domains:
   (a) $f(x) = x^2$, $g(x) = x + 1$
   (b) $f(x) = \sqrt{x}$, $g(x) = x - 2$
   (c) $f(x) = \frac{1}{x}$, $g(x) = x^2 + 1$
   (d) $f(x) = \sin(x)$, $g(x) = 2x$

(7) For the following functions f and subsets $A \subseteq \mathrm{dom}(f)$, determine $\sup_A f$, $\inf_A f$, and whether the maximum and minimum are attained on A:
   (a) $f(x) = x^2$, $A = [0, 2]$
   (b) $f(x) = x^2$, $A = (0, 2)$
   (c) $f(x) = x^3 + x^2 + 2x + 1$, $A = [0, 2]$
   (d) $f(x) = \frac{1}{x}$, $A = (0, 1]$
   (e) $f(x) = \frac{1}{x}$, $A = [1, +\infty)$
   (f) $f(x) = e^x$, $A = [-1, 1]$

(8) Using trigonometric identities, simplify the following expressions:
   (a) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}$
   (b) $(\sin(x) + \cos(x))^2 + (\sin(x) - \cos(x))^2$
   (c) $\frac{1-\cos(2x)}{\sin(2x)}$

(9) Prove the following equalities:
   (a) $\tan(x) + \cot(x) = \frac{2}{\sin(2x)}$
   (b) $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

<u>Additional Practice Problems</u>

⋆⋆ These problems are for you to practice ⋆⋆

⋆⋆ They are not part of the homework assignment ⋆⋆

(1) Sketch the graphs of the following functions (choose a relevant part of the graph to sketch):
- (a) $\sin(x + \frac{\pi}{2})$, $\sin(x) + 1$
- (b) $\log_{10}(\frac{2}{x})$, $\ln(-x)$, $\ln(x^{-1})$
- (c) $-\cos(x)$, $\cos(-x)$
- (d) $\tan(x)$, $\tan(2x)$, $\tan(x + \frac{\pi}{4})$, $-\tan(x)$
- (e) $\arcsin(x)$, $\arcsin(2x)$, $\arcsin(-x)$, $\arcsin(x) + \frac{\pi}{2}$
- (f) $\arctan(x)$, $\arctan(2x)$, $\arctan(-x)$, $-\arctan(x)$

(2) Sketch the graphs of the following power functions:
- (a) $x^{1/2}$, $x^{1/3}$, $x^{1/4}$
- (b) x^{-1}, x^{-2}, x^{-3}
- (c) $x^{2/3}$, $x^{3/2}$, $x^{-1/2}$
- (d) $(x-1)^{1/2}$, $(x+2)^{-1}$, $2x^3$

(3) For the following pairs of functions f and g, compute $f \circ g$ and $g \circ f$, and determine their domains:
- (a) $f(x) = e^x$, $g(x) = \ln(x)$
- (b) $f(x) = |x|$, $g(x) = x - 3$
- (c) $f(x) = \sqrt{x}$, $g(x) = x^2$
- (d) $f(x) = \frac{1}{x-1}$, $g(x) = \frac{1}{x}$
- (e) $f(x) = x^{1/3}$, $g(x) = x^3$
- (f) $f(x) = x^2 + 1$, $g(x) = \sqrt{x-1}$

(4) For the following functions f and subsets $A \subseteq \mathrm{dom}(f)$, determine $\sup_A f$, $\inf_A f$, and whether the maximum and minimum are attained on A:
- (a) $f(x) = e^x$, $A = (-1, 1)$
- (b) $f(x) = \sqrt{4 - x^2}$, $A = (-2, 2)$
- (c) $f(x) = \arctan(x)$, $A = \mathbb{R}$
- (d) $f(x) = \frac{1}{1-x}$, $A = [0, 1)$
- (e) $f(x) = x(2 - x)$, $A = (0, 2)$
- (f) $f(x) = x^{1/3}$, $A = [-8, 8]$
- (g) $f(x) = x^{-2}$, $A = [1, +\infty)$

(5) Determine whether the following functions are even, odd, or neither:
- (a) $f(x) = x^4 + x^2$
- (b) $f(x) = x^5 - x^3$
- (c) $f(x) = x^3 + x^2$
- (d) $f(x) = \frac{x}{x^2+1}$
- (e) $f(x) = x^{-2}$
- (f) $f(x) = |x| + x$

(6) Using trigonometric identities, simplify the following expressions:
- (a) $\sin(x + y)\sin(x - y)$
- (b) $\cos^4(x) - \sin^4(x)$
- (c) $\frac{1+\tan^2(x)}{1+\cot^2(x)}$

(7) Prove the following equality:
- (a) $\frac{\sin(x)+\sin(3x)}{\cos(x)+\cos(3x)} = \tan(2x)$

(8) Simplify the following expressions involving powers:
- (a) $(x^3)^{1/2} \cdot x^{-1/2}$
- (b) $\frac{x^{2/3} \cdot x^{1/2}}{x^{-1/6}}$
- (c) $(x^{-2}y^3)^{-1/2}$
- (d) $\left(\frac{x^{1/3}}{x^{-2/3}}\right)^3$