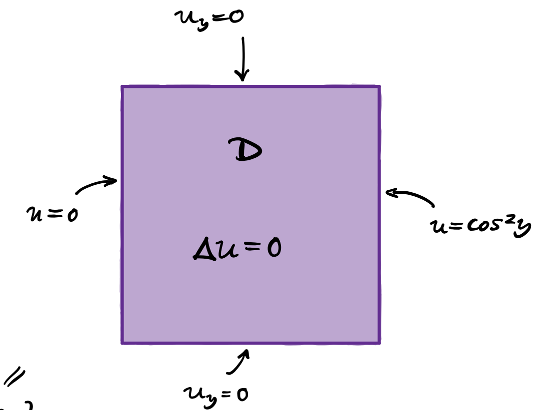
Section 6.2 Q3: Find the harmonic function $u(x,y)$ in the square $D=\{(x,y)\mid 0<x<\pi,\ 0<y<\pi\}$ with the BCs:

$$u_y(x,0)=u_y(x,\pi)=0, \quad u(0,y)=0$$
$$u(\pi,y)=\cos^2 y=\tfrac{1}{2}(1+\cos 2y)$$



We follow the recipe we saw in class:

(i) Separate variables: $u(x,y)=X(x)Y(y)$.

$$0=\Delta u=u_{xx}+u_{yy}=X''(x)Y(y)+X(x)Y''(y)$$
$$\longrightarrow \quad \frac{X''}{X}+\frac{Y''}{Y}=0$$

Since we have homogeneous Neumann BCs in the y variable, we put: $\frac{Y''}{Y}=-\lambda$ and $\frac{X''}{X}=\lambda$.

(ii) We solve:
$$\begin{cases} Y''(y)+\lambda Y(y)=0 & 0<y<\pi \\ Y'(0)=Y'(\pi)=0 \end{cases}$$

This is the homogeneous Neumann problem on the interval $[0,\pi]$. We found the eigenvalues and eigenfunctions in Section 4.2. They are: $\lambda_n=n^2$, $\beta_n=n$ $\quad n=0,1,2,\ldots$ (also 0!)

$$Y_n(y)=\cos(ny).$$

Now the X part:
$$\begin{cases} X''(x)-\lambda X(x)=0 \\ X(0)=0 \end{cases}$$
(the inhomogeneous condition at $x=\pi$ comes later)

For $\lambda>0$ the solutions are $X_n(x)=A\cosh(\beta_n x)+B\sinh(\beta_n x)$

Imposing $X(0)=0$: $X_n(0) = A\cosh(nx) \Longrightarrow A=0.$

So the solutions are $X_n(x) = B_n \sinh(nx).$

For $\lambda = 0$ we have $X_0(x) = Bx + A.$

Imposing $X(0)=0$; $0 = X_0(0) = \longrightarrow A = 0.$

So the solution is $X_0(x) = B_0 x$

(iii) Sum the series:

$$u(x,y) = \frac{B_0 x}{2} + \sum_{n=1}^{\infty} B_n \sinh(nx)\cos(ny)$$

(iv) Impose the inhomogeneous conditions: the only inhomogeneous condition is $u(\pi, y) = \frac{1}{2}(1+\cos 2y).$

$$\frac{1}{2} + \frac{1}{2}\cos(2y) = \frac{B_0 \pi}{2} + \sum_{n=1}^{\infty} B_n \sinh(n\pi)\cos(ny)$$

$$\Longrightarrow \frac{1}{2} = \frac{B_0 \pi}{2} \Longrightarrow B_0 = \frac{1}{\pi}$$

$$\Longrightarrow \frac{1}{2} = B_2 \sinh(n\pi) \Longrightarrow B_2 = \frac{1}{2\sinh(2\pi)}$$

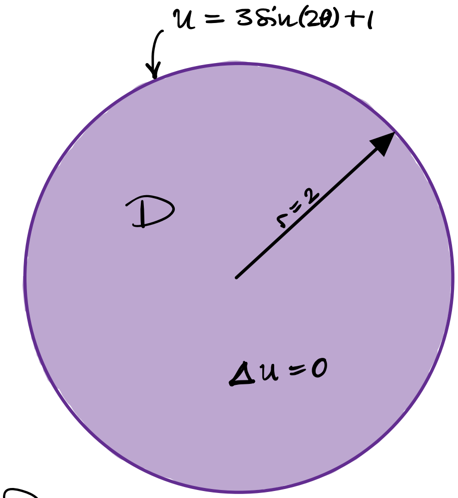
$B_n = 0$ for all $n \neq 0, 2.$

Conclusion: $$u(x,y) = \frac{x}{2\pi} + \frac{\sinh(2x)}{2\sinh(2\pi)}\cos(2y)$$

Section 6.3 Q1: Suppose that u is harmonic in the disk $D=\{r<2\}$ and that $u = 3\sin(2\theta)+1$ for $r=2$. Without finding the solution, answer:

(a) Find the max of u in $\overline{D}$.

(b) Calculate the value of u at the origin.

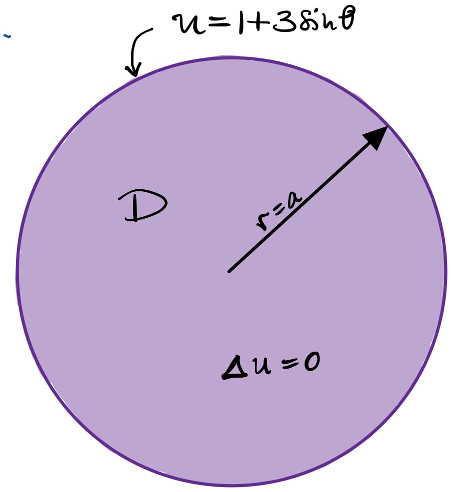


(a) By the strong maximum principle, the max of u is achieved on the boundary $\partial D = \{r=2\}$. So we just need to find the max of $3\sin(2\theta)+1$ over the interval $[0, 2\pi)$. The max is achieved when $\sin(2\theta)=1$, in which case the value is 4.

(b) By the mean value property, for a harmonic function the value at the center of a circle = the average on the circumference. So the value of u at the origin is simply:

$$\frac{1}{2\pi}\int_0^{2\pi} (3\sin(2\theta)+1)\,d\theta = \underbrace{-\frac{3}{4\pi}\cos(2\theta)\Big|_{\theta=0}^{2\pi}}_{0} + 1 = 1$$

Section 6.3 Q2: Solve $u_{xx}+u_{yy}=0$ in the disk $D=\{r<a\}$ with the BC $u=1+3\sin\theta$ on $r=a$.



Poisson's formula gives us the solution

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{1+3\sin\phi}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi$$

but this seems like a difficult integral to solve.
So let's try something else. The fact that u is a constant + a sine on $r=a$ suggests that perhaps we should separate variables and write a Fourier series.

$$0=\Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} = R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)$$

$$\longrightarrow \quad \underbrace{r^2\frac{R''}{R}+r\frac{R'}{R}}_{+\lambda}+\underbrace{\frac{\Theta''}{\Theta}}_{-\lambda}=0$$

$$\begin{cases} \Theta''(\theta)+\lambda\Theta(\theta)=0 \\ r^2R''(r)+rR'(r)-\lambda R(r)=0 \end{cases}$$

We have seen that this leads to: $u(r,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty} r^n\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)$

Then $\quad 1+3\sin\theta = u(a,\theta) = \frac{1}{2}A_0+\sum_{n=1}^{\infty} a^n\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)$

$\longrightarrow \quad 1=\frac{1}{2}A_0, \quad 3=aB_1 \quad \longrightarrow \quad A_0=2, \quad B_1=\frac{3}{a}$, and all other coefficients are 0. So:

$$\boxed{\begin{aligned} u(r,\theta) &= 1+\frac{3}{a}r\sin\theta \\ u(x,y) &= 1+\frac{3y}{a} \end{aligned}}$$

Section 6.3 Q3: Repeat the previous question for the BC

$$u(a,\theta) = \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

Now we have: $$\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

$$\Rightarrow \quad \frac{3}{4} = aB_1, \quad -\frac{1}{4} = a^3 B_3 \quad \Rightarrow \quad B_1 = \frac{3}{4a}, \quad B_3 = -\frac{1}{4a^3}$$

$$u(r,\theta) = \frac{3}{4a} r\sin\theta - \frac{1}{4a^3} r^3 \sin 3\theta$$