Section 5.4 Q2: Consider any series of functions on any finite interval. Show that if it converges uniformly then it also converges in the L^2 sense and in the pointwise sense.

On some interval $[a,b]$, let $\{X_n(x)\}_{n=1}^{\infty}$ and $f(x)$ be functions such that the partial sums $S_N(x) = \sum_{n=1}^{N} A_n X_n(x)$ converge to $f(x)$ uniformly for some constants $\{A_n\}_{n=1}^{\infty}$. That is: we know that
$$\max_{a \le x \le b} |f(x) - S_N(x)| \longrightarrow 0 \quad \text{as } N \to +\infty.$$

(i) Let's start with the simple case: pointwise convergence. For any $y \in (a,b)$,
$$|f(y) - S_N(y)| \le \max_{a \le x \le b} |f(x) - S_N(x)| \longrightarrow 0.$$
This proves pointwise convergence.

(ii) To show L^2-convergence we need to show that
$$\int_a^b |f(x) - S_N(x)|^2 \, dx \quad \text{tends to } 0 \text{ as } N \to +\infty.$$

$$\int_a^b |f(x) - S_N(x)|^2 \, dx \le \int_a^b \underbrace{\max_{a \le x \le b} |f(x) - S_N(x)|^2}_{\text{This is not a function of } x \text{ anymore since we took the 'max'}} dx$$

$$= \max_{a \le x \le b} |f(x) - S_N(x)|^2 \int_a^b dx = \max_{a \le x \le b} |f(x) - S_N(x)|^2 \, (b-a)$$

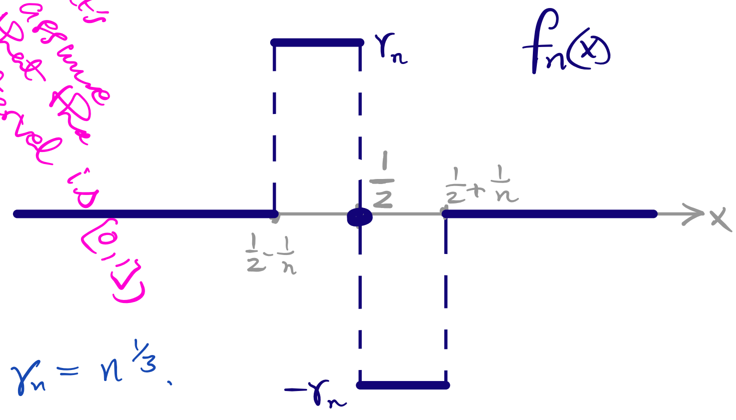
$b-a$ is some constant, and since $\max_{a \le x \le b} |f(x) - S_N(x)| \longrightarrow 0$ in particular eventually $\max_{a \le x \le b} |f(x) - S_N(x)| \le 1$ so that $\max_{a \le x \le b} |f(x) - S_N(x)|^2 \le \max_{a \le x \le b} |f(x) - S_N(x)|$. Therefore
$$\max_{a \le x \le b} |f(x) - S_N(x)|^2 \, (b-a) \longrightarrow 0 \quad \text{as } N \to +\infty.$$

This implies that $\int_a^b |f(x) - S_N(x)|^2 \, dx$ tends to 0 as $N \to +\infty$, i.e. L^2-convergence holds.

Section 5.4 Q3:

Let γ_n be a sequence of constants tending to $+\infty$. Let $f_n(x)$ be the sequence of functions defined as

$$f_n(x) = \begin{cases} \gamma_n & x \in \left[\frac{1}{2}-\frac{1}{n}, \frac{1}{2}\right) \\ 0 & x = \frac{1}{2} \\ -\gamma_n & x \in \left(\frac{1}{2}, \frac{1}{2}+\frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$



(Let's assume that the interval is $[0,1]$)

Show that: (a) $f_n(x) \to 0$ pointwise.

(b) The convergence is not uniform.

(c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.

(d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n = n$.

(a) There are two cases: $x=\frac{1}{2}$ or $x \neq \frac{1}{2}$. If $x = \frac{1}{2}$, then $f_n(\frac{1}{2}) = 0$ $\forall n$.
If $x \neq \frac{1}{2}$, then for $n > \frac{1}{|\frac{1}{2}-x|}$ we have that $|\frac{1}{2}-x| > \frac{1}{n}$ so that $f_n(x) = 0$. Hence, for any x, $f_n(x) \neq 0$ for only finitely many n's, so that $f_n(x) \to 0$ for every x.

(b) For any n, $f_n(\frac{1}{2}+\frac{1}{2n}) = -\gamma_n$ and $f_n(\frac{1}{2}-\frac{1}{2n}) = \gamma_n$ so that

$$\max_{0\le x\le 1} |f_n(x) - 0| \leq |f_n(\tfrac{1}{2}-\tfrac{1}{2n}) - 0| = \gamma_n \longrightarrow +\infty$$

which means there's no uniform convergence.

(c) Suppose that $\gamma_n = n^{1/3}$. Then:

$$\int_0^1 |0 - f_n(x)|^2 dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 \, dx = n^{2/3} \cdot \frac{2}{n} = \frac{2}{n^{1/3}} \xrightarrow{\text{as } n\to\infty} 0$$

so that f_n converges in the L^2 sense.

(d) If $\gamma_n = n$: $\int_0^1 |0 - f_n(x)|^2 dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 dx = n^2 \frac{2}{n} = 2n \xrightarrow{\text{as } n\to\infty} +\infty$

so that f_n does not converge to 0 in the L^2 sense.

Section 5.4 Q12: Start with the Fourier series of $f(x)=x$ on $(0,\ell)$. Apply Parseval's equality. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

In Section 5.1 we worked out both the Fourier sine series and the Fourier cosine series for $f(x)=x$ on $(0,\ell)$. Since $f(x)=x$ is an odd function, it makes more sense to take the Fourier sine series (as sines are also odd).

So we have: $$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell}{n\pi} \sin\left(\frac{n\pi}{\ell} x\right) \quad \circledast$$

Parseval's equality is:
$$\int_a^b |f(x)|^2 dx = \sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx$$

Applying this to $\circledast$ leads to:
$$\int_0^\ell x^2 dx = \sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2} \underbrace{\int_0^\ell \sin^2\left(\frac{n\pi}{\ell}x\right) dx}$$

$$\Longrightarrow \quad \left.\frac{x^3}{3}\right|_{x=0}^{\ell} = \sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2} \frac{1}{2}\ell$$

we saw this in Section 5.1

$$\Longrightarrow \quad \frac{1}{3}\cancel{\ell^3} = \sum_{n=1}^{\infty} \frac{2\cancel{\ell^3}}{n^2\pi^2} \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Section 6.1 Q2: Find the solutions that depend only on r of the eq. $u_{xx}+u_{yy}+u_{zz}=k^2u$, where $k>0$.

Since u is only a function of r, Δu in spherical coordinates becomes: $\Delta u = u_{rr} + \frac{2}{r}u_r \quad \circledast$

Following the hint, let $v = ur$. Then:

$$v_{rr} = (ur)_{rr} = (ru_r + u)_r = ru_{rr} + 2u_r$$

Dividing this by r gives the RHS of $\circledast$, so we have:

$$\frac{v_{rr}}{r} = u_{rr} + \frac{2}{r}u_r = \Delta u = k^2 u$$

$$\Longrightarrow \quad v_{rr} = k^2 ru = k^2 v.$$

The solutions of $v_{rr} = k^2 v$ are

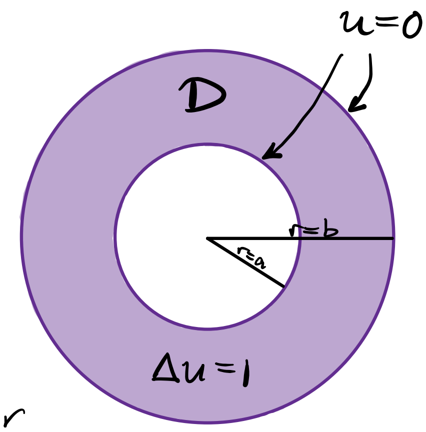
$$v(r) = A\cosh(kr) + B\sinh(kr)$$

$$\Longrightarrow \quad u(r) = \frac{A}{r}\cosh(kr) + \frac{B}{r}\sinh(kr)$$

Section 6.1 Q6: Solve $u_{xx}+u_{yy}=1$ in the annulus $a<r<b$ with u vanishing on both ends.

We need to solve:

$$\begin{cases} \Delta u = u_{rr} + \frac{1}{r}u_r = 1 & a<r<b \\ u(a) = u(b) = 0 \end{cases}$$



We have seen that $u_{rr} + \frac{1}{r}u_r = \frac{1}{r}(ru_r)_r$

$\Longrightarrow \quad \frac{1}{r}(ru_r)_r = 1 \quad \longrightarrow \quad (ru_r)_r = r$

$\Longrightarrow \quad ru_r = \frac{1}{2}r^2 + c_1 \quad \longrightarrow \quad u_r = \frac{1}{2}r + \frac{c_1}{r}$

$\longrightarrow \quad u(r) = \frac{1}{4}r^2 + c_1 \ln r + c_2$

Now we apply the BCs: $\quad 0 = u(a) = \frac{a^2}{4} + c_1 \ln a + c_2$

$\quad 0 = u(b) = \frac{b^2}{4} + c_1 \ln b + c_2$

Subtracting these we have: $\quad c_1 = \dfrac{b^2 - a^2}{4(\ln a - \ln b)}$

which allows us to find: $\quad c_2 = \dfrac{a^2 \ln b - b^2 \ln a}{4(\ln a - \ln b)}$.

$$\Longrightarrow \quad u(r) = \frac{1}{4}r^2 + \frac{b^2 - a^2}{4(\ln a - \ln b)}\ln r + \frac{a^2 \ln b - b^2 \ln a}{4(\ln a - \ln b)}$$

Section 6.1 Q9: A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at 100°C. The outer bdry satisfies $u_r = -\gamma < 0$, where $\gamma = $ const.

(a) Find the temp.

(b) What are the hottest and coldest temperatures?

(c) Can you choose γ so that the temp. on the outer boundary is 20°C?

As a steady-state of the heat/diffusion eq, we have the simple eq. $\Delta u = 0$. Written in 3D spherical coordinates for a function that only depends on r, this becomes:

with BCs:
$$\begin{cases} \Delta u = u_{rr} + \frac{2}{r} u_r = 0 & 1 < r < 2 \\ u(1) = 100 \\ u_r(2) = -\gamma \end{cases}$$

We know that the solution of the eq. is $u(r) = -\frac{c_1}{r} + c_2$.

Plug in the BC at $r=1$: $100 = -\frac{c_1}{1} + c_2 = c_2 - c_1$

BC at $r=2$: $-\gamma = u_r(2) = \frac{c_1}{2^2} = \frac{1}{4} c_1 \Rightarrow c_1 = -4\gamma$

and $c_2 = 100 + c_1 = 100 - 4\gamma$.

(a) $$u(r) = \frac{4\gamma}{r} + 100 - 4\gamma.$$

(b) In (a) we found a function that is decreasing in r, so that the hottest temp is $u(1) = 100$.
The coldest: $u(2) = 100 - 2r$.

(c) Choose $r = 40$.