Section 5.1 Q2: Let $\phi(x) = x^2$ for $0 \leq x \leq l = 1$.

(a) Calculate its Fourier sine series.

(b) Calculate its Fourier cosine series.

(a) The Fourier sine series is $\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right)$.
The coefficients are given by: $A_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx$.
Here $\phi(x) = x^2$ and $l = 1$:

$$A_n = 2\int_0^1 x^2 \sin(n\pi x)\, dx =$$

int. by parts (twice)

$$= 2\int_0^1 2x \frac{1}{n\pi} \cos(n\pi x)\, dx - 2\underbrace{\left[x^2 \frac{1}{n\pi} \cos(n\pi x)\right]_{x=0}^{1}}_{\frac{1}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n} - 0}$$

$$= -2\underbrace{\int_0^1 2 \frac{1}{n^2\pi^2} \sin(n\pi x)\, dx + 2\underbrace{\left[\frac{2x}{n^2\pi^2} \sin(n\pi x)\right]_{x=0}^{1}}_{\frac{2}{n^2\pi^2}\underbrace{\sin(n\pi)}_{0} - 0}}_{0} - \frac{2}{n\pi}(-1)^n$$

$$= \underbrace{\left[\frac{4}{n^3\pi^3} \cos(n\pi x)\right]_{x=0}^{1}}_{\frac{1}{n^3\pi^3}(\underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\cos 0}_{1})} - \frac{2}{n\pi}(-1)^n$$

$$= \frac{4}{n^3\pi^3}\left((-1)^n - 1\right) - \frac{2}{n\pi}(-1)^n$$

$$\Rightarrow \quad x^2 = \sum_{n=1}^{\infty} \left[\frac{4}{n^3\pi^3}\left((-1)^n - 1\right) - \frac{2}{n\pi}(-1)^n\right] \sin(n\pi x)$$

(b) For the cosine series we have:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

$$A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{n\pi}{\ell}x\right)dx$$

with $\phi(x) = x^2$ and $\ell = 1$ we have:

$(n \geq 1)$

$$A_n = 2\int_0^1 x^2 \cos(n\pi x)\,dx$$

$$= -2\int_0^1 2x \frac{1}{n\pi}\sin(n\pi x)\,dx + \underbrace{2\left[x^2 \frac{1}{n\pi}\sin(n\pi x)\right]_{x=0}^{1}}_{\underbrace{\frac{1}{n\pi}\underbrace{\sin(n\pi)}_{0} - 0}_{0}}$$

$$= -4\int_0^1 \frac{1}{n^2\pi^2}\cos(n\pi x)\,dx$$

$$+ \underbrace{\left[4x \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_{x=0}^{1}}_{\frac{4}{n^2\pi^2}\left[\underbrace{\cos(n\pi)}_{(-1)^n} - 0\right]}$$

$$= \underbrace{\left[-\frac{4}{n^3\pi^3}\sin(n\pi x)\right]_{x=0}^{1}}_{\underbrace{-\frac{4}{n^3\pi^3}\left(\underbrace{\sin n\pi}_{0} - \underbrace{\sin 0}_{0}\right)}_{0}} + (-1)^n \frac{4}{n^2\pi^2} = (-1)^n \frac{4}{n^2\pi^2}$$

$$A_0 = 2\int_0^1 x^2\,dx = \frac{2}{3}x^3\Big|_{x=0}^{1} = \frac{2}{3}$$

$$\Longrightarrow \quad x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2\pi^2}\cos(n\pi x)$$

Section 5.1 Q9: Solve

Neumann BCs!

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0<x<\pi \quad t>0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t>0 \\ u(x,0) = 0 & 0<x<\pi \\ u_t(x,0) = \cos^2 x & 0<x<\pi \end{cases}$$

We know that the solution of the wave eq. with Neumann BCs is given by:

$$u(x,t) = \tfrac{1}{2}A_0 + \tfrac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left[A_n \cos\left(\tfrac{n\pi}{\ell}ct\right) + B_n \sin\left(\tfrac{n\pi}{\ell}ct\right)\right]\cos\left(\tfrac{n\pi}{\ell}x\right)$$

Plug in $\ell = \pi$ to get:

$$u(x,t) = \tfrac{1}{2}A_0 + \tfrac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left[A_n \cos(nct) + B_n \sin(nct)\right]\cos(nx)$$

$$u_t(x,t) = \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} nc\left[-A_n \sin(nct) + B_n \cos(nct)\right]\cos(nx)$$

$$0 = u(x,0) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \underbrace{\cos 0}_{1} + B_n \underbrace{\sin 0}_{0}\Big]\cos(nx) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) \qquad (1)$$

$$\cos^2 x = u_t(x,0) = \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} nc\Big[-A_n \underbrace{\sin 0}_{0} + B_n \underbrace{\cos 0}_{1}\Big]\cos(nx) \qquad (2)$$

(1) $\Rightarrow$ $0 = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$ $\Rightarrow$ all A_n are 0.

(2) $\Rightarrow$ $\cos^2 x = \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} \underbrace{ncB_n}_{\tilde{B}_n}\cos(nx)$

This last expression is a Fourier cosine series for $\cos^2 x$!

We need to compute the coefficients:

$$B_0 = \frac{2}{\pi}\int_0^\pi \cos^2 x \, dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$$

$(n \geq 1)$

$$\tilde{B}_n = \frac{2}{\pi}\int_0^\pi \cos^2 x \cos(nx) \, dx = \frac{1}{\pi}\int_0^\pi (1+\cos(2x))\cos(nx)\, dx$$

$$= \frac{1}{\pi}\underbrace{\int_0^\pi \cos(nx)\, dx}_{0} + \underbrace{\frac{1}{\pi}\int_0^\pi \cos(2x)\cos(nx)\, dx}_{\text{the only time this is nonzero is when } n=2 \text{ because of the orthogonality of the cosines}}$$

$$\Rightarrow \tilde{B}_2 = \frac{1}{\pi}\underbrace{\int_0^\pi \cos^2(2x)\, dx}_{\frac{\pi}{2}} = \frac{1}{2} \Rightarrow B_2 = \frac{\tilde{B}_2}{2c} = \frac{1}{4c}$$

$$\tilde{B}_n = 0 \quad \forall n \neq 0, 2$$

Conclusion: $B_0 = 1$, $B_2 = \frac{1}{4c}$

and all other B's and A's $= 0$.

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\sin(2ct)\cos(2x)$$