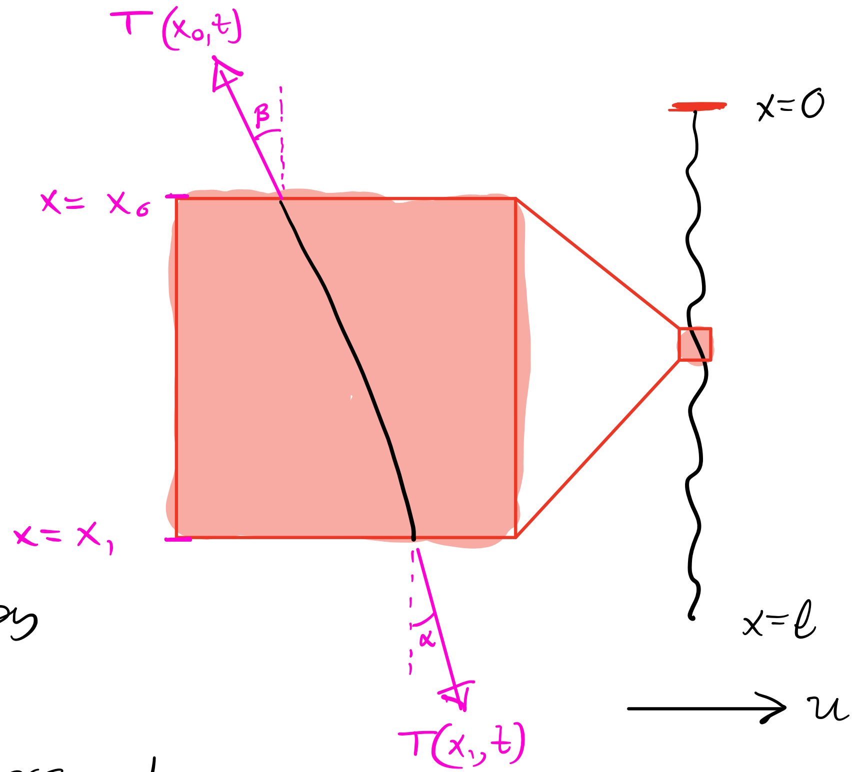
Section 1.3 Q2: A flexible chain of length ℓ is hanging from one end $x=0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.



Let ρ be the density of the chain, which is assumed to be constant.

For any point x, the mass below it is given by

$$(\ell-x)\rho$$

and the gravitational force by

$$(\ell-x)\rho g$$

where g is the acceleration due to gravity. This force acts tangentially. So the horizontal part is given by $(\ell-x)\rho g \frac{u_x}{\sqrt{1+u_x^2}}$.

Assuming small oscillations means that

$\sqrt{1+u_x^2} \approx 1$, so that the horizontal part is given by $(\ell-x)\rho g u_x$.

So we find that in the horizontal direction for the segment $[x_0, x_1]$ we have:

$$\underbrace{(l-x_1)\rho g u_x(x_1,t) - (l-x_0)\rho g u_x(x_0,t)}_{\text{LHS}} = F_{hor} = m a_{hor} = \underbrace{\int_{x_0}^{x_1} \rho u_{tt}(x,t)\,dx}_{\text{RHS}}$$

Replace $x_1 = x_0 + h$ and divide by h to get:

$$\text{LHS} = \frac{(l-(x_0+h))\rho g u_x(x_0+h,t) - (l-x_0)\rho g u_x(x_0,t)}{h}$$

$$\text{RHS} = \frac{1}{h}\int_{x_0}^{x_0+h} \rho u_{tt}(x,t)\,dx$$

In the limit $h \to 0$ we find:

$$\text{LHS} = ((l-x)\rho g u_x)_x$$

$$\text{RHS} = \rho u_{tt}$$

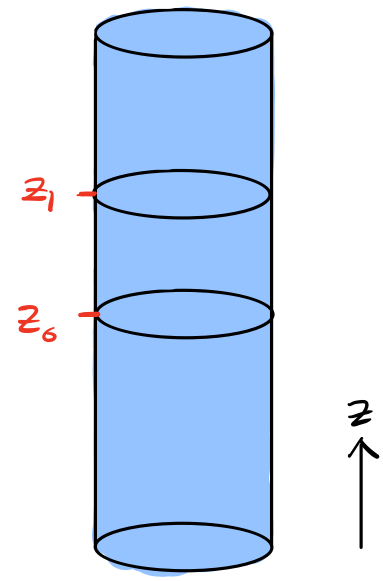
So we eventually have: $u_{tt} = g((l-x)u_x)_x$

Note that here we did not need to investigate the vertical force since it is already given to us, unlike in the string example from class.

Section 1.3 Q4:

Suppose that some particles which are suspended in a liquid medium would be pulled down at constant velocity $V > 0$ by gravity in the absence of diffusion. Taking account of diffusion, find the equation for the concentration of particles. Assume homogeneity in the horizontal directions x and y. Let the z axis point upwards.

Since the concentration is homogeneous in x, y, we can look at a cylinder pointing in the z direction like this



The mass of dye between z_0 and z_1 is given by:

$$M(t) = \int_{z_0}^{z_1} u(z,t)\, dz$$

(note that this is function of time).

Hence $$\frac{dM}{dt} = \int_{z_0}^{z_1} u_t(z,t)\, dz$$

is the rate of change of dye in the volume between z_0 and z_1.

This must equal the difference between dye entering this region, and dye exiting:

$$\text{flow in} = k u_z(z_1, t) + V u(z_1, t)$$

$$\text{flow out} = k u_z(z_0, t) + V u(z_0, t)$$

So we have

$$\int_{z_0}^{z_1} u_t(z,t)\, dz = k u_z(z_1,t) - k u_z(z_0,t) + V u(z_1,t) - V u(z_0,t)$$

Replacing $z_1 = z_0 + h$, and dividing by h, we have:

$$\frac{1}{h}\int_{z_0}^{z_0+h} u_t(z,t)\, dz = k \frac{u_z(z_0+h,t) - u_z(z_0,t)}{h} + V \frac{u(z_0+h,t) - u(z_0,t)}{h}$$

Letting $h \to 0$ we finally have:

$$u_t = k u_{zz} + V u_z .$$

Remark: How did we discover that the flow in at z_1 has the additional term of $V u(z_1,t)$ (and similarly for z_0)?

Consider two times $t_0 < t_1$. How much dye will ented through z_1 during this time? All the dye between z_1 and $z_1 + V(t_1 - t_0)$:

$$z_1 + V(t_1 - t_0) - z_1 = V(t_1 - t_0)$$

And what is the rate at which the dye enters? We need to normalize the above amount by dividing by the amount of time:

$$\frac{V(t_1 - t_0)}{t_1 - t_0} = V.$$

Thus V is the rate, and must be multiplied by concentration u.