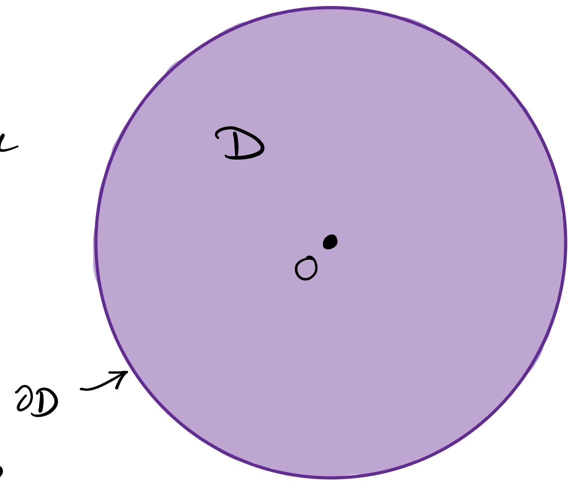
Theorem: (Mean Value Property)

Let u be a harmonic function in a disk D and continuous on $\bar{D} = D \cup \partial D$. Then the value of u at the center of D equals the average of u on its circumference ∂D.



Proof: Without loss of generality, assume that the center of D is at $(x,y) = (0,0)$. From Poisson's formula we know that

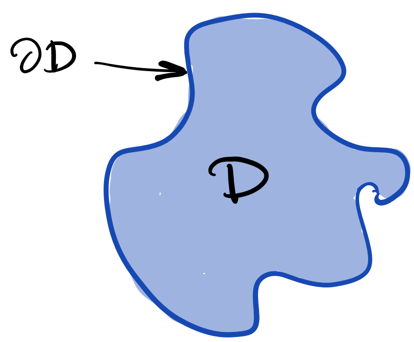
$$u(r=0) = \frac{a^2}{2\pi} \int_0^{2\pi} \frac{u(\phi)}{a^2}\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\, d\phi$$

which is, by definition, the average of u on ∂D. $\square$

Theorem: (Strong Maximum Principle)

Let D be a connected and bounded open set in $\mathbb{R}^2$. Let $u(x,y)$ be harmonic in D and continuous in $\bar{D} = D \cup \partial D$. Then the max and min of u are attained on ∂D and nowhere inside D (unless u is a constant function).

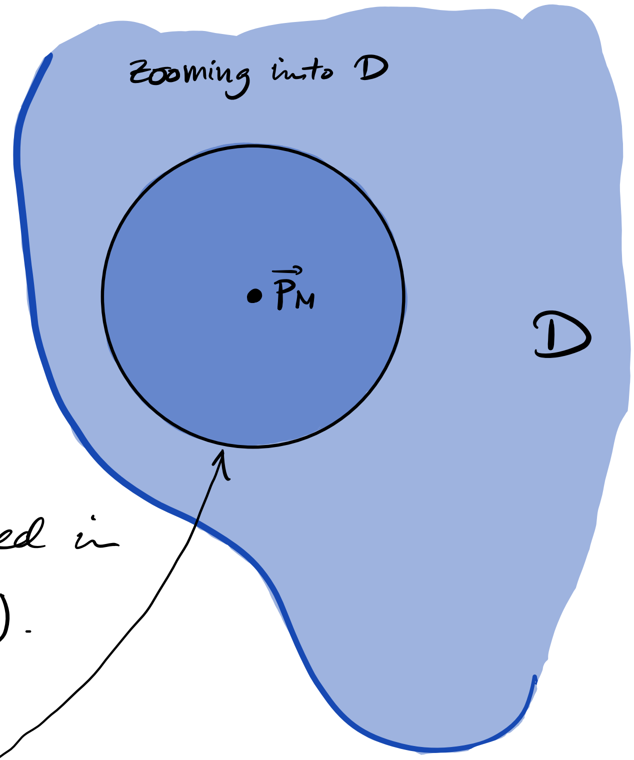
Proof: We've seen (in Section 6.1) a proof of the weak version. Now we can prove the strong version.



Suppose there's a point $\vec{p}_M$ in D where the max of u, call it M, is achieved.

That is: $\quad u(\vec{p}) \leq u(\vec{p}_M) = M$

for any $\vec{p} \in D$.



Draw a circle around $\vec{p}_M$ that is contained in D (we can do this because D is open).

Now we use the mean value property:

$$M = u(\vec{p}_M) = \text{average on circle}$$

However, the average cannot be more than the max M. So we have $M = u(\vec{p}_M) = \text{average on circle} \leq M$

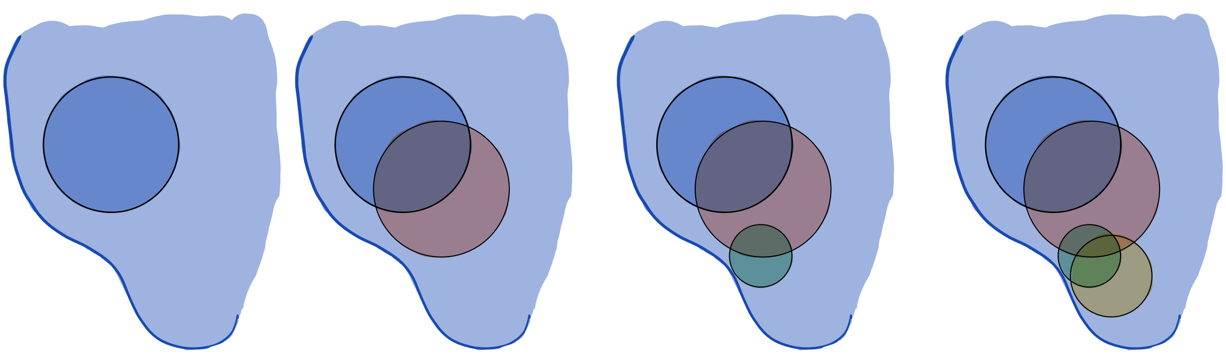
$\longrightarrow$ average on the circle $= M$

If there are points on the circle where $u < M$, there must be other points where $u > M$. But this would contradict M being the max.

$\longrightarrow$ $u = M$ on the entire circle

But we could have chosen the circle to be of any radius (so long as it is $\subset D$). So $u = M$ on the entire blue shaded disk. Now we can repeat the argument starting from any other point in the blue disk to get the red disk. Eventually, we can reach every point in D.

(here we use the fact that D is bounded and connected)



dot, dot, dot!

Conclusion: $u \equiv M$ everywhere in D.

I.e.: if u attains its max inside of D, u must be constant. Otherwise, the max can only be attained on ∂D.

Similarly for the min. $\square$

A slightly more rigorous proof:

Suppose the u attains its max M at some point $\vec{p}_M \in D$. Let $\vec{p} \in D$ be any other point. Let Γ be a curve contained in D linking $\vec{p}_M$ and $\vec{p}$. Let $d > 0$ be the distance between Γ and ∂D (d is positive since both $\vec{p}_M, \vec{p}$ are in D, Γ is chosen to be in D and D itself is open). Let B_1 be a disk centered at $\vec{p}_M$ with radius $\frac{d}{2}$. Then $B_1 \subset D$. By the mean value property,

$$M = u(\vec{p}_M) = \text{average of } u \text{ on } \partial B_1$$

The average of u on any set cannot exceed M. So we have: $M = \text{average on } \partial B_1 \leq M$. Hence the average must be $= M$. The value of u cannot exceed M at any point on ∂B_1; so, in order for the average to be M, the value of u also cannot be $< M$ at any point. Hence $u = M$ on ∂B_1.

The same argument can be repeated for any disk of radius $\alpha\frac{d}{2}$ around $\vec{p}_M$, for any $\alpha \in (0,1)$. Hence $u = M$ on the entire disk B_1.

Now, choose a point $\vec{p}_1 \in \Gamma \cap \partial B_1$. $u(\vec{p}_1) = M$.
Let B_2 be a disk of radius $\frac{d}{2}$ centered at $\vec{p}_1$. By the same argument as before (applied to $\vec{p}_1$ instead of $\vec{p}_M$), $u \equiv M$ on B_2.
Chose a point $\vec{p}_2 \in \Gamma \cap \partial B_2$ and repeat these arguments.

Important point: since Γ is a closed curve, and all disks B_n are of a fixed radius, only finitely many are required to cover Γ.

Conclusion: $u(\vec{p}) = M$.

However, $\vec{p}$ was arbitrary.

So $u = M$ everywhere in D

Conclusion: if the max is attained in D, then u is simply constant. Otherwise, the max would necessarily have to be on ∂D.