6.1 Lalace's Equation

We have so far dealt primarily with the wave eq. $u_{tt} = c^2 u_{xx}$ and the diffusion eq. $u_t = k u_{xx}$. Now, we think of what happens for large times when (perhaps) the solution has settled to some steady-state. Then, if everything is steady we expect that

$$u_t = 0 \quad \text{and} \quad u_{tt} = 0.$$

In both cases, this leaves us with $u_{xx} = 0$. This is called Laplace's equation, which we shall consider in higher dimensions too:

$$\text{1D:} \quad u_{xx}(x) = 0$$

$$\text{2D:} \quad \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 0$$

$$\text{3D:} \quad \Delta u(x,y,z) = u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0.$$

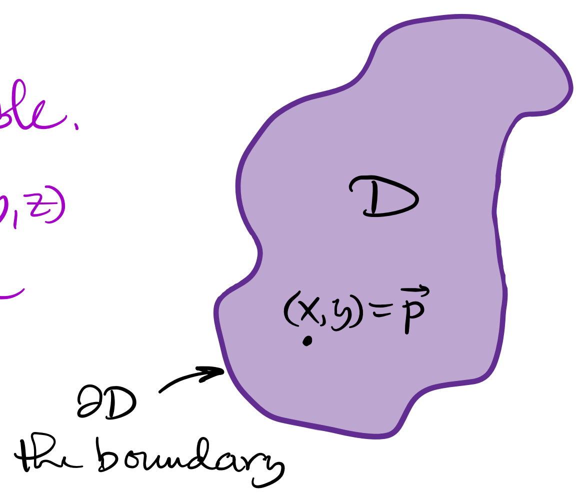
The operator $\mathcal{L}u = \Delta u$ is called the Laplacian.
Any solution is called a harmonic function.

We consider higher dimensions since 1D is rather boring: solutions have the form $u(x) = Ax + B$.

If the right hand side is nonzero: $\Delta u = f$ the eq. is called Poisson's eq.

So, now we won't have a time variable.
Just spatial variables (x,y) or (x,y,z) which will typically belong to some open set D in $\mathbb{R}^2$ or $\mathbb{R}^3$.

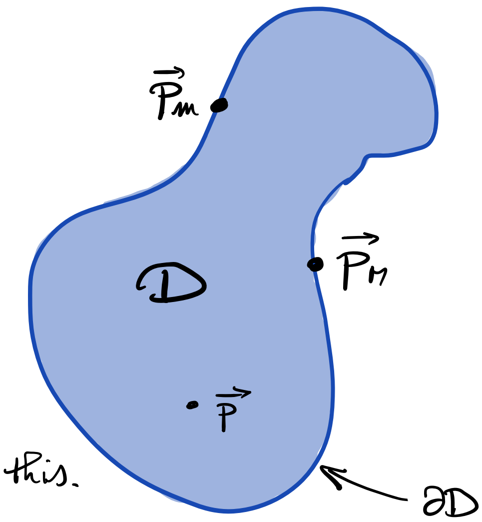
("open" means that D doesn't include its boundary)



Laplace's eq. satisfies a max. principle, much like the diffusion eq.
It holds in any dimension, we stick to 2D for simplicity.

Theorem: (Maximum Principle)

Let $D \subset \mathbb{R}^2$ be a connected open set. Let ∂D be its boundary, and $\overline{D} = D \cup \partial D$ its closure. Let $u(x,y)$ be a harmonic function in D that is continuous in $\overline{D}$. Then the maximum and minimum values of u in $\overline{D}$ are attained in ∂D and nowhere in D (unless u is a constant function).



Proof: We need to prove that:

(1) $\exists\ \vec{p}_m = (x_m, y_m)$ and $\vec{p}_M = (x_M, y_M)$ in ∂D

s.t. $\quad u(\vec{p}_m) \leq u(\vec{p}) \leq u(\vec{p}_M)$

$\forall\ \vec{p} = (x,y) \in D$.

(2) There are no points in D that satisfy this.

I.e. there are no $\vec{q}_m, \vec{q}_M \in D$ s.t.

$u(\vec{q}_m) \leq u(\vec{p}) \leq u(\vec{q}_M) \quad \forall\ \vec{p} = (x,y) \in D$.

We prove (1) now (similar to the proof we've seen before) and (2) will be proven later.

Let $\varepsilon > 0$ and define $v(\vec{p}) = u(\vec{p}) + \varepsilon |\vec{p}|^2$ (where we recall that $|\vec{p}|^2 = x^2 + y^2$). Then:

$$\Delta v = \Delta u + \varepsilon \Delta (|\vec{p}|^2) = \Delta u + \varepsilon (\partial_{xx} + \partial_{yy})(x^2 + y^2)$$
$$= \Delta u + \varepsilon (2+2) = \underbrace{\Delta u}_{\substack{=0 \\ \text{since } u \\ \text{is harmonic}}} + 4\varepsilon = 4\varepsilon > 0$$

At a local max $\vec{p} \in D$, $\Delta v(\vec{p}) = \partial_{xx} v(\vec{p}) + \partial_{yy} v(\vec{p}) \leq 0$, so there's no local max of v inside D. Since $v(x,y)$ is continuous in the <u>closed</u> set $\overline{D}$, it must attain its max (and min) there, and, in particular, these must be attained in ∂D (the boundary). Suppose $v(\vec{p})$ attains its max at $\vec{p}_0 \in \partial D$. So $\forall \vec{p} \in \overline{D}$:

$$u(\vec{p}) \leq v(\vec{p}) \leq v(\vec{p}_0) = \underbrace{u(\vec{p}_0)} + \varepsilon \underbrace{|\vec{p}_0|^2}$$

This is smaller than $\max\limits_{\vec{q} \in \partial D} u(\vec{q})$

This is smaller than the square of the distance of the point in ∂D which is farthest from the origin. Call this distance l.

$$\Rightarrow \quad u(\vec{p}) \leq \max_{\vec{q} \in \partial D} u(\vec{q}) + \varepsilon l^2$$

This is true for any $\varepsilon > 0$ (as small as we wish) so it must still hold true for $\varepsilon = 0$, i.e. $u(\vec{p}) \leq \max\limits_{\vec{q} \in \partial D} u(\vec{q})$, $\forall \vec{p} \in \overline{D}$.

Since ∂D is a closed set, $\max\limits_{\vec{q} \in \partial D} u(\vec{q})$ is attained at <u>some</u> point on the boundary, call it $\vec{p}_M \in \partial D$. So:

$$\forall \vec{p} \in \overline{D}, \qquad u(\vec{p}) \leq \max_{\vec{q} \in \partial D} u(\vec{q}) = u(\vec{p}_M)$$

Similarly, there exists $\vec{p}_m \in \partial D$ s.t. $u(\vec{p}_m) \leq u(\vec{p}) \quad \forall \vec{p} \in \overline{D}$.

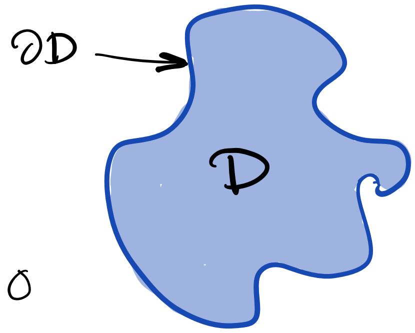
$\square$

(Proof of (2) will follow later).

Theorem: (Uniqueness of Solutions)

The Dirichlet problem for Poisson's eq. $\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$

has a unique solution.



Proof: Suppose that there exist solutions u and v. Let $w = u - v$.

Then: $\Delta w = \Delta(u-v) = \Delta u - \Delta v = f - f = 0$

and $w = u - v = h - h = 0$ on ∂D.

So w solves the problem: $\begin{cases} \Delta w = 0 & \text{in } D \\ w = 0 & \text{on } \partial D \end{cases}$

From the maximum principle we know that the max of w in $\overline{D}$ is achieved on ∂D. But w is 0 there. Similarly for the min of w. So $w \equiv 0$ everywhere.

$0 \equiv w = u - v \implies u \equiv v$ everywhere.

$\square$

Invariance under rigid transformations:

Proposition:

The Laplacian is unaffected by translations and rotations.

Proof is an exercise (it is in the book, 2D: p. 156, 3D: p. 158).

This means that moving D or rotating it won't change the result. It also suggests to look at radial/spherical symmetry.

The Laplacian in 2D in polar coordinates:

Let $x = r\cos\theta$, $y = r\sin\theta$.

The Jacobian is: $J = \begin{pmatrix} \partial_r x & \partial_r y \\ \partial_\theta x & \partial_\theta y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$

and the Jacobian for the inverse transformation is

$$J^{-1} = \begin{pmatrix} \partial_x r & \partial_x \theta \\ \partial_y r & \partial_y \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}.$$

$$\Longrightarrow \quad \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

To compute $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ we need to proceed with caution, because we can't just square these expressions. There will be cross-terms. It's best to try it on a function:

$$\frac{\partial^2}{\partial x^2} f = \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right] f$$
$$= \cos^2\theta \frac{\partial^2}{\partial r^2} f - \cos\theta \frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} f\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r} f\right)$$
$$+ \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} f\right)$$
$$= \cos^2\theta \frac{\partial^2}{\partial r^2} f + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} f - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} f + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} f$$
$$- \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} f + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} f + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} f$$
$$= \left[\cos^2\theta \frac{\partial^2}{\partial r^2} - 2\frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\right] f$$

Similarly: $\frac{\partial^2}{\partial y^2} f =$

$$= \left[\sin^2\theta \frac{\partial^2}{\partial r^2} + 2\frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}\right] f$$

Hence we get: $\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$

$$= \left[(\cos^2\theta + \sin^2\theta)\frac{\partial}{\partial r^2} + \frac{\sin^2\theta + \cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\theta+\cos^2\theta}{r}\frac{\partial}{\partial r}\right] f$$

Using the fact that $\sin^2\theta + \cos^2\theta = 1$ we conclude that

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

For functions that are radially symmetric (i.e. do not depend upon θ) all θ derivatives vanish, so the operator becomes:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$ (for functions that are radially symmetric)

We can easily solve the homogeneous problem coming from this:

$$\underbrace{u_{rr} + \frac{1}{r}u_r}_{\frac{1}{r}(ru_r)_r} = 0.$$

check this: $(ru_r)_r = (r_r)u_r + ru_{rr} = u_r + ru_{rr}$

$$\frac{1}{r}(ru_r)_r = 0 \iff (ru_r)_r = 0 \iff ru_r = c_1$$

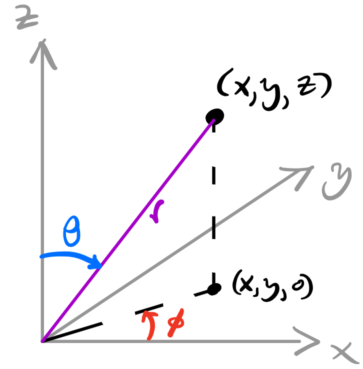
$$\iff u_r = \frac{c_1}{r}$$

$$\iff u(r) = c_1 \ln r + c_2$$

$\ln r = \ln\left(\sqrt{x^2+y^2}\right)$ is an extremely important harmonic function in 2D.

The Laplacian in 3D in spherical coordinates:

We use the convention that ϕ is the angle in the (x,y) plane (azimuthal angle) and θ is the angle from the z-axis (polar angle) to be consistent with the book.



We skip the details of the derivation of the formula for Δ in spherical coordinates (try it for yourselves!) and write the final expression:

$$\boxed{\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$

In the case of a radial function (i.e. no θ or ϕ dependence) this reduces to

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

Solutions of this are as follows: $\underbrace{u_{rr} + \frac{2}{r}u_r}_{\frac{1}{r^2}(r^2 u_r)_r} = 0$

$$\frac{1}{r^2}(r^2 u_r)_r = 0 \iff (r^2 u_r)_r = 0 \iff r^2 u_r = c_1$$

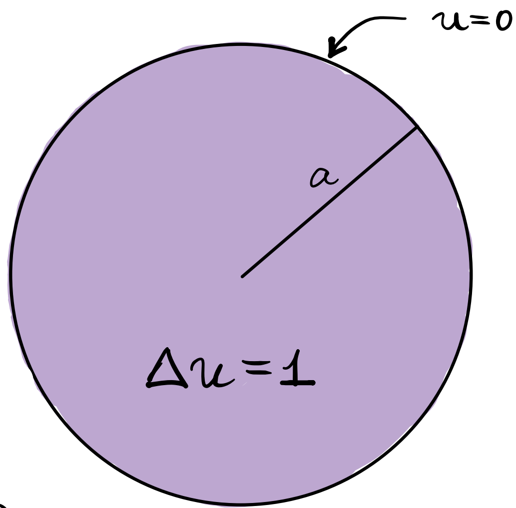
$$\iff u_r = \frac{c_1}{r^2}$$

$$\iff \boxed{u(r) = -\frac{c_1}{r} + c_2}$$

$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is an extremely important harmonic function in 3D.

Example: (Section 6.1 Q5)

Solve $u_{xx}+u_{yy}=1$ in $r<a$ with $u(x,y)=0$ on $r=a$.



In 2D we know that the Laplacian takes the form
$$\Delta u = u_{rr}+\frac{1}{r}u_r$$
for radial functions. So our problem becomes:
$$\begin{cases} u_{rr}(r)+\frac{1}{r}u_r(r)=1 & \text{in } (0,a) \\ u(a)=0 \end{cases}$$

We've seen that $u_{rr}+\frac{1}{r}u_r=\frac{1}{r}(ru_r)_r$ so that our equation is reduced to $\frac{1}{r}(ru_r)_r=1 \Longleftrightarrow (ru_r)_r=r$

$\Longleftrightarrow \quad ru_r=\frac{r^2}{2}+c_1 \quad \Longleftrightarrow \quad u_r=\frac{r}{2}+\frac{c_1}{r}$

$\Longleftrightarrow \quad u(r)=\frac{r^2}{4}+c_1\ln r+c_2.$

Now we impose the boundary conditions. Notice that we only have the condition $u(a)=0$. However notice another important aspect: our domain D includes $r=0$ (the origin) where $\ln r$ is <u>not defined</u>. So we need that term to <u>vanish</u>,

$\rightarrow$ we require $c_1=0$. Hence we're left with $u(r)=\frac{r^2}{4}+c_2$. Imposing $u(a)=0$ leads to

$c_2=-\frac{a^2}{4} \quad \longrightarrow \quad u(r)=\frac{1}{4}(r^2-a^2).$