# 5.3 Orthogonality and General Fourier Series

We now consider some general properties of Fourier series.

NOTE THAT EVERYTHING HERE IS COMPLEX-VALUED!

Consider the general interval: $(a,b)$

Let $f(x)$, $g(x)$ be function on $(a,b)$ (they can be complex-valued).

Define an inner-product (a.k.a. dot product) for $f, g$ as:

$$(f,g) := \int_a^b f(x)\overline{g(x)}\,dx$$

where an overline $\overline{\phantom{x}}$ means the complex conjugate.

We say that $f, g$ are <u>orthogonal</u> if $(f,g)=0$.

Recall the operators and boundary conditions:

$\mathcal{L}_D$ = (negative) second derivative operator with Dirichlet BCs:

(D) $$\mathcal{L}_D f(x) = -f''(x), \quad f(a)=f(b)=0$$

$\mathcal{L}_N$ = (negative) second derivative operator with Neumann BCs:

(N) $$\mathcal{L}_N f(x) = -f''(x), \quad f'(a)=f'(b)=0.$$

$\mathcal{L}_P$ = (negative) second derivative operator with periodic BCs:

(P) $$\mathcal{L}_N f(x) = -f''(x), \quad f'(a)=f'(b),\ f(a)=f(b)$$

This last BC is a new one, corresponding to the full Fourier series

**Green's Second Identity:** Take two functions $y_1(x)$, $y_2(x)$ on $(a,b)$.

Then:
$$(-y_1'\bar{y}_2 + y_1\bar{y}_2')' = -y_1''\bar{y}_2 - \cancel{y_1'\bar{y}_2'} + \cancel{y_1'\bar{y}_2'} + y_1\bar{y}_2''$$
$$= -y_1''\bar{y}_2 + y_1\bar{y}_2''.$$

We can integrate and use the fundamental thm. of calculus to get:
$$\left.(-y_1'\bar{y}_2 + y_1\bar{y}_2')\right|_{x=a}^{b} = \int_a^b (-y_1''\bar{y}_2 + y_1\bar{y}_2'')\,dx \qquad \circledast$$

This is called Green's Second Identity.

**Lemma:** Assume that both $y_1$, $y_2$ satisfy either Dirichlet, or Neumann or periodic BCs. Then the LHS of $\circledast$ is 0.

**Proof:** Let's check for Dirichlet (check Neumann yourself!)

$$\text{LHS of } \circledast = -y_1'(b)\underbrace{\bar{y}_2(b)}_{0} + \underbrace{y_1(b)}_{0}y_2'(b) - \left(-y_1'(a)\underbrace{\bar{y}_2(a)}_{0} + \underbrace{y_1(a)}_{0}y_2'(a)\right)$$
$$= 0$$

Let's check the periodic case:

$$\text{LHS of } \circledast = \underbrace{-y_1'(b)\bar{y}_2(b)}_{\mathrm{I}} + \underbrace{y_1(b)y_2'(b)}_{\mathrm{II}} - \left(\underbrace{-y_1'(a)\bar{y}_2(a)}_{\mathrm{I}} + \underbrace{y_1(a)y_2'(a)}_{\mathrm{II}}\right)$$

The two terms I are equal (with opposite signs), as are the two terms II. So we get 0.

$\square$

**Observation:** Let $\mathcal{L}$ be one of $\mathcal{L}_D$, $\mathcal{L}_N$ or $\mathcal{L}_P$.

Suppose that $(\lambda, X)$ are an eigenvalue – eigenfunction pair:

$\mathcal{L}X = \lambda X$. Then:

$$\mathcal{L}\overline{X} = \overline{\mathcal{L}}\,\overline{X} = \overline{\mathcal{L}X} = \overline{\lambda X} = \lambda^* \overline{X}$$

$\longrightarrow$ $(\lambda^*, \overline{X})$ are also an eigenvalue - eigenfunction pair.

**Theorem:** In all three cases ( (D), (N), or (P) )

there are <u>no</u> complex eigenvalues and any eigenfunction can be taken to be real-valued.

**Proof:** In Green's Second Identity $\circledast$ replace $y_1, y_2$ with some function $X(x)$. Then

$$\left(-X'\overline{X} + X\overline{X}'\right)\Big|_{x=a}^{b} = \int_a^b \left(-X''\overline{X} + X\overline{X}''\right) dx$$

Now suppose that $X$ is an eigenfunction of $\mathcal{L}_D$, $\mathcal{L}_N$ or $\mathcal{L}_P$ with eigenvalue $\lambda$.

From the Lemma we know that the LHS $= 0$. Hence:

$$0 = \int_a^b \left(-X''\overline{X} + X\overline{X}''\right) dx = \int_a^b \left(\lambda X\overline{X} - X\lambda^*\overline{X}\right) dx$$

$$= (\lambda - \lambda^*) \int_a^b X(x)\overline{X}(x)\, dx = (\lambda - \lambda^*) \int_a^b |X(x)|^2 dx$$

Since $|X(x)|^2 \geq 0$ and since $X(x)$ is not trivially 0, the integral $\int_a^b |X(x)|^2 dx$ is strictly positive (why?). Therefore we must have $\lambda - \lambda^* = 0$ which can only be true if $\lambda \in \mathbb{R}$.

We need to show that $X(x)$ can be taken to be real-valued. Suppose that $X(x)$ is complex-valued and write it as $X(x) = Y(x) + i\,Z(x)$ where $Y, Z$ are real-valued. Then:

$$-Y''(x) - i\,Z''(x) = -X''(x) = \lambda X(x) = \lambda Y(x) + i\lambda Z(x)$$

Taking real and imaginary parts we have:

$$-Y''(x) = \lambda Y(x) \qquad -Z''(x) = \lambda Z(x)$$

We know that $X$ satisfies (D), (N), or (P). $Y$ and $Z$ will satisfy the same BCs as well (check this!).

So $Y, Z$ are real-valued eigenfunctions satisfying the same BCs as $X$. Since $\overline{X}$ has eigenvalue $\lambda^* = \lambda$ (eigenvalues are real!) we conclude that we can replace $X, \overline{X}$ by $Y, Z$, observing that $\text{span}\{X, \overline{X}\} = \text{span}\{Y, Z\}$.

So we have shown that $\lambda$ can be taken with eigenfunctions $Y$ and $Z$ (which are both real) rather than $X, \overline{X}$. $\square$

**Theorem:** In all three cases ((D), (N), or (P)), any two eigenfunctions corresponding to different eigenvalues are orthogonal.

Proof: We already know that all eigenvalues are real, and that eigenfunctions can be taken to be real-valued. Take $X_1(x)$ and $X_2(x)$ corresponding to two different eigenvalues $\lambda_1, \lambda_2$.

In Green's Second Identity, replace $y_1, y_2$ by $X_1, X_2$. The LHS is 0 from the Lemma.

So we have:

$$
\begin{aligned}
0 &= \int_a^b \left(-X_1'' X_2 + X_1 X_2''\right) dx \\
&= \int_a^b \left(\lambda_1 X_1 X_2 + X_1(-\lambda_2 X_2)\right) dx \\
&= (\lambda_1 - \lambda_2) \int_a^b X_1 X_2 \, dx
\end{aligned}
$$

Since $\lambda_1 \neq \lambda_2$ (by assumption) this means that

$$
\int_a^b X_1 X_2 \, dx = 0 \implies (X_1, X_2) = 0
$$

$\square$

**Theorem:** In all three cases ((D), (N), or (P)), there are no negative eigenvalues.

Proof: The starting point is Green's First Identity:

$$\boxed{\int_a^b f''(x)\, g(x)\, dx = -\int_a^b f'(x)\, g'(x)\, dx + f'(x)\, g(x) \Big|_{x=a}^{b}}$$

(this is just integration by parts)

Choose $f, g$ to be a (real) eigenfunction $X(x)$ with a (real) eigenvalue $\lambda$. Then we have:

$$LHS = \int_a^b X'' X \, dx = \int_a^b -\lambda X(x) X(x)\, dx = -\lambda \int_a^b X^2\, dx$$

$$RHS = -\int_a^b X' X' \, dx + X' X \Big|_{x=a}^{b}$$

$$= -\int_a^b (X')^2\, dx + \underbrace{X'(b) X(b) - X'(a) X(a)}_{=0 \text{ for (D), (N), (P)}}$$

$$\Rightarrow \quad \lambda \int_a^b X^2\, dx = \int_a^b (X')^2\, dx$$

$$\Rightarrow \quad \lambda = \frac{\int_a^b (X'(x))^2\, dx}{\int_a^b (X(x))^2\, dx} \geq 0$$

(in fact, we get $\lambda = 0$ iff $X(x) = \text{const} \neq 0$ which is only possible for (N), (P), but not (D))

∎

**Theorem:** In all three cases ((D), (N), or (P)), there are infinitely many eigenvalues tending to $+\infty$:

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \longrightarrow +\infty$$

Corresponding to $\lambda_n$ is an eigenfunction $X_n(x)$ which can be chosen to be real, and orthogonal to all other eigenfunctions.

We take this theorem without proof.