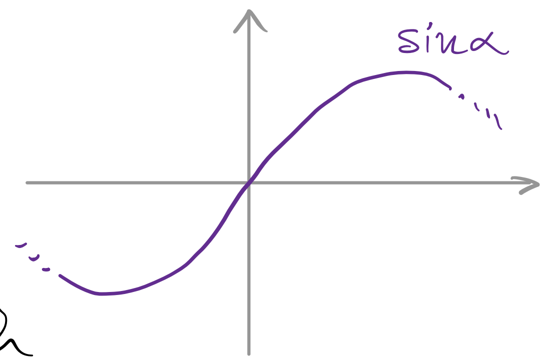
5.2 Even, Odd, Periodic and Complex Functions

Fourier Sine Series: $\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right)$ on $(0,\ell)$.

Observe that $\sin(\alpha)$ is an odd function, i.e.

$$\sin(-\alpha) = -\sin\alpha$$

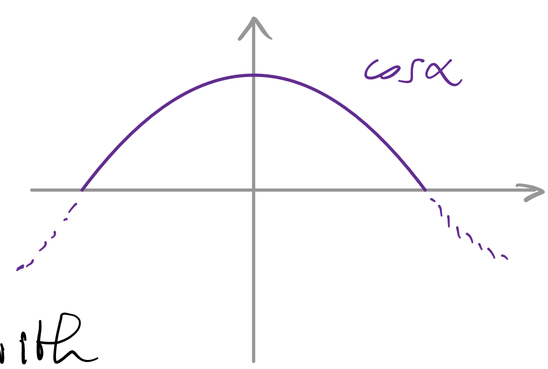


This means that if we extend $\phi(x)$ to $(-\ell,0)$, we will get an odd extension since each of the sines making up $\phi(x)$ are odd. That is, an extension with sines will give $\phi(x)$ on $(-\ell,\ell)$ with $\phi(-x) = -\phi(x)$.

Fourier Cosine Series: $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right)$ on $(0,\ell)$

$\cos(\alpha)$ is an even function:

$$\cos(-\alpha) = \cos\alpha$$



So if we extend $\phi(x)$ to $(-\ell,0)$ with cosines, we'll get an even extension. That is, we'll get $\phi(x)$ on $(-\ell,\ell)$ with $\phi(-x) = \phi(x)$.

Full Fourier Series:

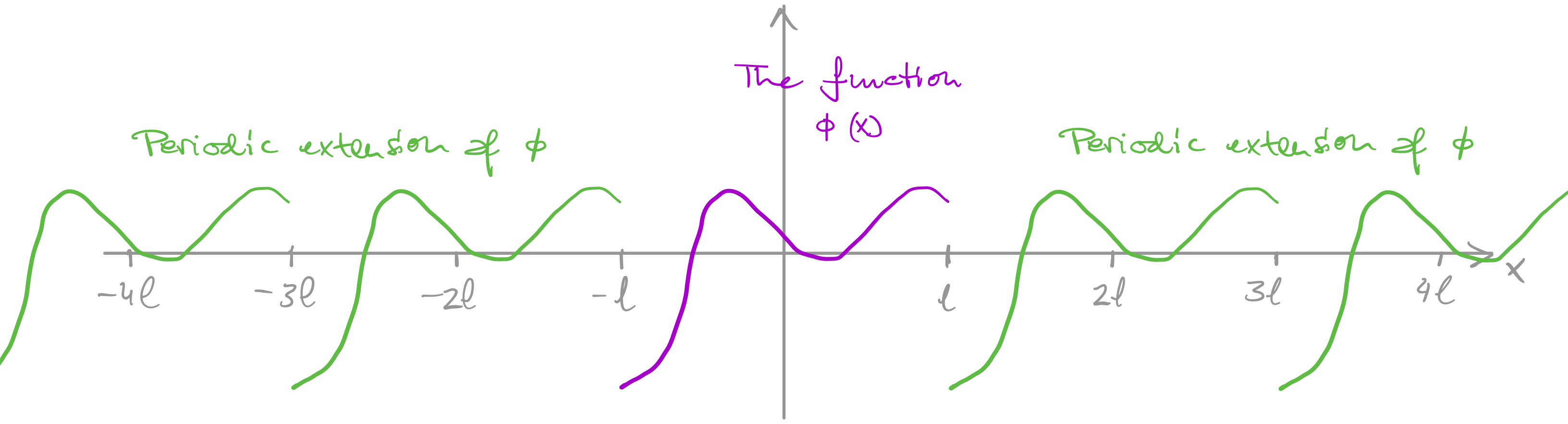
$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right) + B_n \sin\left(\frac{n\pi}{\ell}x\right) \quad \text{on } (-\ell,\ell)$$

Observe that $\cos\theta$ and $\sin\theta$ have period of 2π:

$$\cos\theta = \cos(\theta+2\pi k) \qquad \sin\theta = \sin(\theta+2\pi k) \qquad k\in\mathbb{Z}.$$

Therefore
$$\cos\left(\frac{\pi}{\ell}nx\right) = \cos\left(\frac{\pi}{\ell}nx+2\pi k\right) = \cos\left(\frac{\pi}{\ell}(nx+2\ell k)\right)$$
$$\sin\left(\frac{\pi}{\ell}nx\right) = \sin\left(\frac{\pi}{\ell}nx+2\pi k\right) = \sin\left(\frac{\pi}{\ell}(nx+2\ell k)\right) \qquad k\in\mathbb{Z}$$
have periods of 2ℓ.

$\Longrightarrow$ If we extend $\phi(x)$ outside of $(-\ell,\ell)$ it will extend periodically:



So we can think of the full Fourier series of $\phi(x)$ either as an expansion in sines and cosines of ϕ on $(-\ell,\ell)$ or as an expansion of the periodic extension of ϕ on $\mathbb{R}$.

Complex Form of the Full Fourier Series:

Using the DeMoivre formulas

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

to replace the basis $\{1, \cos(\frac{\pi}{\ell}x), \sin(\frac{\pi}{\ell}x), \cos(\frac{2\pi}{\ell}x), \sin(\frac{2\pi}{\ell}x), \dots\}$

with: $\{1, e^{i\frac{\pi}{\ell}x}, e^{-i\frac{\pi}{\ell}x}, e^{i\frac{2\pi}{\ell}x}, e^{-i\frac{2\pi}{\ell}x}, \dots\}$

which can simply be written as: $\{e^{i\frac{n\pi}{\ell}x}\}_{n=-\infty}^{\infty}$

$$\Longrightarrow \quad \phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{\ell}x}$$

Lemma: $$\int_{-\ell}^{\ell} e^{i\frac{n\pi}{\ell}x} e^{-i\frac{m\pi}{\ell}x}\,dx = \begin{cases} 0 & n \neq m \\ 2\ell & n = m \end{cases}$$

Proof: $$\int_{-\ell}^{\ell} e^{i\frac{n\pi}{\ell}x} e^{-i\frac{m\pi}{\ell}x}\,dx = \int_{-\ell}^{\ell} e^{i(n-m)\frac{\pi}{\ell}x}\,dx =: I$$

if $n\neq m$: $$I = \frac{1}{i(n-m)\frac{\pi}{\ell}} e^{i(n-m)\frac{\pi}{\ell}x}\Big|_{x=-\ell}^{\ell}$$

$$= \frac{\ell}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right]$$

$$= \frac{\ell}{i(n-m)\pi}\left[(-1)^{n-m} - (-1)^{-(n-m)}\right] = 0$$

if $n=m$: $I = \int_{-\ell}^{\ell} 1\,dx = 2\ell$. $\square$

Since these exponentials are orthogonal, we can identify the coefficients c_n as we have done before (inspired from the finite-dimensional case):

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \phi(x)\, e^{-i\frac{n\pi}{\ell}x}\, dx \qquad n = 0, \pm 1, \pm 2, \pm 3, \dots$$