4.2 The Neumann Condition

We now consider:

The Neumann Condition = Specifying the Value of u_x on the Boundary

---

We now consider either the wave equation:

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0<x<l \quad t>0 \\ u_x(0,t) = u_x(l,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) & 0<x<l \end{cases}$$

or the diffusion equation:

$$\begin{cases} u_t(x,t) = k u_{xx}(x,t) & 0<x<l \quad t>0 \\ u_x(0,t) = u_x(l,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) & 0<x<l \end{cases}$$

Notice that now the boundary conditions involve u_x rather than u!

Using separation of variables $u(x,t) = X(x)T(t)$ we reach the same equations for X and T as before.

X part: As before, we have $X''(x) + \beta^2 X(x) = 0$ which leads to solutions of the form:

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

Let's write the derivative of this, which we will need:

$$X'(x) = -C\beta\sin(\beta x) + D\beta\cos(\beta x).$$

$$u_x(0,t) = 0 \implies X'(0) = 0 \implies -C\beta\underbrace{\sin 0}_{0} + D\beta\underbrace{\cos 0}_{1} = 0$$

$$\implies D\beta = 0 \implies D = 0.$$

$$u_x(\ell,t) = 0 \implies X'(\ell) = 0 \implies -C\beta\sin(\beta\ell) = 0$$

$$\implies \beta\ell = n\pi \implies \beta_n = \frac{n\pi}{\ell}, \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n = 1, 2, \ldots$$

$$X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right)$$

These are the EIGENFUNCTIONS for the Neumann problem

Can we have eigenvalues that are not positive?
I.e. can we solve $-X''(x) = \lambda X(x) \quad 0<x<\ell$
with the boundary cond: $X'(0) = X'(\ell) = 0$
and with $\lambda \in \mathbb{C}$ which is not positive?

Try $\lambda = 0$: we get $X''(x) = 0$ so that
$$X(x) = C + Dx, \quad X'(x) = D$$
Apply BCs: $0 = X'(0) = X'(\ell) = D$.
$\longrightarrow$ We can satisfy the BCs with $D = 0$.
$\longrightarrow$ $X(x) = C$ (constant)
is a legitimate solution!
$\Longrightarrow$ $\lambda = 0$ is an eigenvalue!

$\lambda < 0$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$: It can be shown that such values of λ cannot be eigenvalues, but we skip that for now.

So the eigenvalues are:
$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n = 0, 1, 2, \ldots$$

These are the EIGENVALUES
for the Neumann problem

T part: The $T(t)$ part will be identical to what we saw before, with the exception of the part coming from $\lambda = 0$.

Diffusion equation: For $\lambda_n \neq 0$ we again have:

$$T'(t) = -\lambda_n k T(t)$$
$$\longrightarrow \quad T(t) = Ae^{-\lambda_n k t}$$

For $\lambda = 0$ we have $T'(t) = 0 \implies T(t) = A$.

So, for $n = 1, 2, 3, \ldots$ we have as before:

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \cos\left(\frac{n\pi}{\ell}x\right) \qquad n = 1, 2, \ldots$$

Notice that the sine is now a cosine!

And we also have a u_0 now: the spatial part is $\cos\left(\frac{0 \cdot \pi}{\ell}x\right) = 1$ and the temporal part is a constant which we called A above. For reasons which will become clear, we call $A_0 = 2A$, to find:

$$u(x,t) = \frac{1}{2}A_0 + \sum_n A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \cos\left(\frac{n\pi}{\ell}x\right)$$

In addition, the initial condition will have to satisfy:

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

Wave equation: For $\lambda > 0$ we get the same behavior as we've seen before, so we have:

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right]\cos\left(\frac{n\pi}{\ell}x\right)$$

For $\lambda = 0$ we get $X_0(x) = \text{const}$ as before. For the T part we have $T''(t) = \underset{0}{\underbrace{\lambda}} c^2 T(t) = 0$ so that $T_0(t) = A + Bt$. This T_0 term goes with the X_0 term which is a constant. So, to conclude, the general solution has the form:

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_n \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right]\cos\left(\frac{n\pi}{\ell}x\right)$$

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

$$\psi(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_n \frac{n\pi}{\ell} c B_n \cos\left(\frac{n\pi}{\ell}x\right)$$