The Diffusion Equation:

The analogous problem for the diffusion eq. is:

$$\begin{cases} u_t(x,t) = k\, u_{xx}(x,t) & 0<x<\ell \quad t>0 \\ u(0,t) = u(\ell,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) & 0<x<\ell \end{cases}$$

Following the same procedure, we make the ansatz:

$$u(x,t) = X(x)\,T(t).$$

This leads to: $\quad X(x)\,T'(t) = k\, X''(x)\,T(t).$

Dividing by kXT we get

$$-\frac{T'}{kT} = -\frac{X''}{X} = \lambda$$

as before (where we have inserted a $-$ sign since we anticipate λ to be positive).

Notice that the temporal part only has a T', not T'': this is going to be crucial!

X part: As before, we have $-X''(x) = \lambda X(x)$ with the boundary conditions $X(0) = X(\ell) = 0$. Exactly as before, we get multiples of $\sin(\frac{n\pi}{\ell}x)$.

T part: The T part is: $T'(t) = -\lambda k\, T(t)$.

The general solution is $T(t) = A e^{-\lambda k t}$

The X part requires discrete values: $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ as before.

So we find the sequence of solutions:

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\left(\frac{n\pi}{\ell}x\right)$$

And the general solution is any (finite) sum:

$$u(x,t) = \sum_n A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\left(\frac{n\pi}{\ell}x\right)$$

and to satisfy the initial condition we need:

$$\phi(x) = u(x,0) = \sum_n A_n e^{-0} \sin\left(\frac{n\pi}{\ell}x\right) = \sum_n A_n \sin\left(\frac{n\pi}{\ell}x\right)$$

Comparing temporal behavior:

The temporal parts of the solutions contain:

- $A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)$ WAVE
- $A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$ DIFFUSION

This again demonstrates that solutions to the wave eq. conserve energy (sin and cos are periodic) while solutions to the diffusion eq. have decaying energy (decaying exponential).

Fourier Series:

The expression of a function as a sum of sines and cosines is called a Fourier series (Joseph Fourier 1768–1830).

It turns out that we can take *infinite* sums (but with caution!).

This is an extremely important concept in all the sciences.

This can be done for almost any function.

When we wrote $\phi(x) = \sum_n A_n \sin\left(\frac{n\pi}{\ell} x\right)$, then the RHS is called the Fourier expansion of ϕ. Since there are only sines, the series is called a Fourier sine series.

Below we'll have an example with a series of cosines. That will be called a Fourier cosine series.

Comparison to Linear Algebra:

Let's look again at the equation $X''(x) = -\lambda X(x)$.
Define $\mathcal{L}$ to be the operator that sends a function to $-$ its second derivative: $\mathcal{L}(X(x)) = -X''(x)$.
Then this equation becomes:

$$\mathcal{L}X = \lambda X.$$

Compare this to $Ax = \lambda x$ which you have seen in linear algebra (where A is an $n \times n$ matrix and x is a vector)

This is the same!

In linear algebra, a solution x is called an eigen<u>vector</u> and the corresponding number λ is called an eigen<u>value</u>.

Here, we discovered that there are <u>infinitely</u> many solutions

$$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right) \quad ; \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n = 1, 2, 3, \ldots$$

These are called: EIGENFUNCTIONS EIGENVALUES

They depend on the boundary conditions. Below, for the Neumann problem, we'll see that the eigenfunctions are cosines, not sines.

Let us justify the assumption that all eigenvalues λ_n are positive numbers:

Proposition: All eigenvalues of $\begin{cases} -X''(x) = \lambda X(x) & 0<x<l \\ X(0) = X(l) = 0 \end{cases}$

are positive.

Proof: Let us rule out any other option.

- Could $\lambda = 0$ be an eigenvalue?

  Suppose so. Then we would have $X''(x) = 0$

  $\Longrightarrow$ $X(x) = Dx + C$ for some constants C, D.

  Plugging in $X(0) = 0 \Rightarrow C = 0$

  Then $X(l) = 0 \Rightarrow Dl = 0 \Rightarrow D = 0$.

  So $\lambda = 0$ cannot be an eigenvalue.

- Could $\lambda < 0$ be an eigenvalue?

  Suppose so. Then write it as $\lambda = -\gamma^2$ for some $\gamma \in \mathbb{R}$. Then $X'' = \gamma^2 X$

  $\longrightarrow$ $X(x) = C\cosh(\gamma x) + D\sinh(\gamma x)$.

  $0 = X(0) = C$, $0 = X(l) = D\sinh(\gamma l)$. Since $\gamma l \neq 0$ we have $D = 0$.

  So $\lambda < 0$ cannot be an eigenvalue.

- Could λ complex be an eigenvalue?

  Suppose so. Denote $\sqrt{-\lambda} = \pm\gamma$, where $\gamma \in \mathbb{C}$.
  The equation $X''(x) = -\lambda X(x) = \gamma^2 X(x)$ has solutions of the form $X(x) = Ce^{\gamma x} + De^{-\gamma x}$ (where this is the complex exponential function).

  $$0 = X(0) = C + D \Rightarrow C = -D$$

  $$0 = X(\ell) = Ce^{\gamma\ell} + De^{-\gamma\ell} = -De^{\gamma\ell} + De^{-\gamma\ell}$$

  $$= D\left(e^{-\gamma\ell} - e^{\gamma\ell}\right)$$

  $$\Rightarrow e^{-\gamma\ell} - e^{\gamma\ell} = 0 \Rightarrow e^{-\gamma\ell} = e^{\gamma\ell}$$

  $$\Rightarrow e^{2\gamma\ell} = 1$$

  By properties of the complex exponential function this implies that $2\gamma\ell$ is a <u>purely</u> complex number and that $\mathrm{Im}(2\gamma\ell) = 2\pi n$, $n \in \mathbb{Z}$.

  $$\Rightarrow 2\pi\gamma i = 2\pi n, \ n \in \mathbb{Z} \Rightarrow \gamma = -i\frac{n\pi}{\ell}, \ n \in \mathbb{Z}$$

  $$\Rightarrow \lambda = -\gamma^2 = \left(\frac{n\pi}{\ell}\right)^2.$$ But this is real and positive!

So the only eigenvalues are the positive numbers

$$\left(\frac{\pi}{\ell}\right)^2, \ \left(\frac{2\pi}{\ell}\right)^2, \ \left(\frac{3\pi}{\ell}\right)^2, \ \ldots$$

4.2 The Neumann Condition

We now consider:

The Neumann Condition = Specifying the Value of u' on the Boundary

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We now consider either the wave equation:

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0<x<l \quad t>0 \\ u_x(0,t) = u_x(l,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) & 0<x<l \end{cases}$$

or the diffusion equation:

$$\begin{cases} u_t(x,t) = k u_{xx}(x,t) & 0<x<l \quad t>0 \\ u_x(0,t) = u_x(l,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) & 0<x<l \end{cases}$$

Notice that now the boundary conditions involve u_x rather than u!

Using separation of variables $u(x,t) = X(x)T(t)$ we reach the same equations for X and T as before.

X part: As before, we have $X''(x)+\beta^2 X(x)=0$ which leads to solutions of the form:

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

Let's write the derivative of this, which we will need:

$$X'(x) = -C\beta\sin(\beta x) + D\beta\cos(\beta x).$$

$$u_x(0,t)=0 \implies X'(0)=0 \implies -C\beta\underbrace{\sin 0}_{0} + D\beta\underbrace{\cos 0}_{1} = 0$$

$$\implies D\beta = 0 \implies D=0.$$

$$u_x(\ell,t)=0 \implies X'(\ell)=0 \implies -C\beta\sin(\beta\ell)=0$$

$$\implies \beta\ell = n\pi \implies \beta_n = \frac{n\pi}{\ell}, \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n=1,2,\dots$$

$$X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right)$$

These are the EIGENFUNCTIONS for the Neumann problem

Can we have eigenvalues that are not positive?
I.e. can we solve $-X''(x) = \lambda X(x) \quad 0<x<\ell$
with the boundary cond: $X'(0) = X'(\ell) = 0$
and with $\lambda \in \mathbb{C}$ which is not positive?

Try $\lambda = 0$: we get $X''(x) = 0$ so that
$X(x) = C + Dx, \quad X'(x) = D$

Apply BCs: $0 = X'(0) = X'(\ell) = D$.

$\longrightarrow$ We can satisfy the BCs with $D=0$.

$\longrightarrow$ $X(x) = C$ (constant)

is a legitimate solution!

$\Longrightarrow$ $\lambda = 0$ is an eigenvalue!

$\lambda < 0$ or $\lambda \in \mathbb{C}\setminus\mathbb{R}$: It can be shown that such values of λ cannot be eigenvalues but we skip that for now.

So the eigenvalues are:

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n = 0, 1, 2, \ldots$$

These are the EIGENVALUES for the Neumann problem

T part: The $T(t)$ part will be identical to what we saw before, with the exception of the part coming from $\lambda = 0$.

Diffusion equation: For $\lambda_n \neq 0$ we again have:

$$T'(t) = -\lambda_n k T(t)$$
$$\rightarrow \quad T(t) = A e^{-\lambda_n k t}$$

For $\lambda = 0$ we have $T'(t) = 0 \implies T(t) = A$.

So, for $n = 1, 2, 3, \ldots$ we have as before:

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \cos\left(\frac{n\pi}{\ell} x\right) \qquad n = 1, 2, \ldots$$

Notice that the sine is now a cosine!

And we also have a u_0 now: the spatial part is $\cos\left(\frac{0 \cdot \pi}{\ell} x\right) = 1$ and the temporal part is a constant which we called A above. For reasons which will become clear, we call $A_0 = 2A$, to find:

$$u(x,t) = \frac{1}{2} A_0 + \sum_n A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \cos\left(\frac{n\pi}{\ell} x\right)$$

In addition, the initial condition will have to satisfy:

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

Wave equation: For $\lambda > 0$ we get the same behavior as we've seen before, so we have:

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right] \cos\left(\frac{n\pi}{\ell}x\right)$$

For $\lambda = 0$ we get $X_0(x) = \text{const}$ as before. For the T part we have $T''(t) = \underset{0}{\underbrace{\lambda}} c^2 T(t) = 0$ so that $T_0(t) = A + Bt$. This T_0 term goes with the X_0 term which is a constant. So, to conclude, the general solution has the form:

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_n \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right] \cos\left(\frac{n\pi}{\ell}x\right)$$

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

$$\psi(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_n \frac{n\pi}{\ell} c B_n \cos\left(\frac{n\pi}{\ell}x\right)$$