4.1 The Dirichlet Condition on an Interval

As we have seen in Section 1.4 there are different types of boundary conditions. In this section we dig deeper into:

The Dirichlet Condition = Specifying the Value of u on the Boundary

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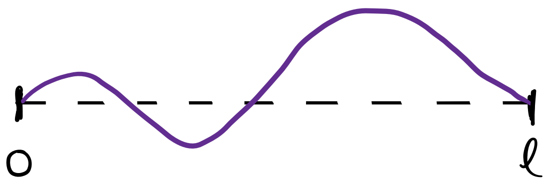
The Wave Equation:



We consider a string, fixed at its two ends at $x=0$, $x=\ell$

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0<x<\ell \quad t>0 \\ u(0,t) = u(\ell,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) & 0<x<\ell \end{cases}$$

with some initial conditions.

We try to solve by making an ansatz (= educated guess) that the solution can be separated into a part depending on x and a part depending on t:

$$u(x,t) = X(x)\,T(t).$$

Plugging this into the wave eq. we have:

$$X(x)\,T''(t) = u_{tt} = c^2 u_{xx} = c^2 X''(x)\,T(t)$$

Dividing by $-c^2 XT$ this becomes:

$$-\frac{1}{c^2}\frac{T''}{T} = -\frac{X''}{X}$$

The LHS is only a function of t, and the RHS is only a function of x. The only way for them to equal one another is if they are both constant. We call this constant λ. So

$$-\frac{1}{c^2}\frac{T''}{T} = -\frac{X''}{X} = \lambda$$

(we will see that λ must be positive, which is why we chose to introduce a $-$ sign in front of $\frac{1}{c^2}\frac{T''}{T}$ and in front of $\frac{X''}{X}$)

$\hookrightarrow$ Since λ will be positive, there exists $\beta \in \mathbb{R}$ such that $\beta^2 = \lambda$. So we replace λ by β^2.

X part: We start with the equation $-\frac{X''}{X} = \beta^2$.

$$\Rightarrow \quad X''(x) + \beta^2 X(x) = 0$$

We know how to solve this: sines and cosines!

$$\Rightarrow \quad X(x) = C\cos(\beta x) + D\sin(\beta x)$$

where C, D are constants.

Now we impose the boundary conditions: the string is fixed at $x = 0, \ell$, so that

$$X(0) = 0 \quad \Rightarrow \quad C\underbrace{\cos 0}_{1} + D\underbrace{\sin 0}_{0} = 0 \quad \Rightarrow \quad C = 0$$

$$X(\ell) = 0 \quad \Rightarrow \quad D\underbrace{\sin(\beta\ell)}_{\text{this must be } 0} = 0$$

In order for $\sin(\beta\ell)$ to be 0, we must have $\beta\ell = n\pi$. There are infinitely many β's that satisfy this:

$$\beta_n = \frac{n\pi}{\ell}$$

so that:

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad (n = 1, 2, 3, \ldots)$$

and $X_n(x)$ is a multiple of $\sin\left(\frac{n\pi}{\ell}x\right)$

T part: The T part is: $T'' + c^2\beta^2 T = 0$

$$\Longrightarrow \quad T(t) = A\cos(\beta c t) + B\sin(\beta c t)$$

where A, B are constants.

Recalling that $u(x,t) = X(x)T(t)$ we find that for each n we have a solution of the form:

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right] \sin\left(\frac{n\pi}{\ell}x\right)$$

By linearity, we can sum finitely many such solutions u_n, so that

$$u(x,t) = \sum_n \left[A_n \cos\left(\frac{n\pi}{\ell}ct\right) + B_n \sin\left(\frac{n\pi}{\ell}ct\right)\right] \sin\left(\frac{n\pi}{\ell}x\right)$$

is a solution of the wave eq. that satisfies $u(0,t) = u(\ell,t) = 0$. To satisfy the initial conditions we must have:

$$\phi(x) = u(x,0) = \sum_n \left[A_n \underbrace{\cos\left(\frac{n\pi}{\ell}c\cdot 0\right)}_{1} + B_n \underbrace{\sin\left(\frac{n\pi}{\ell}c\cdot 0\right)}_{0}\right] \sin\left(\frac{n\pi}{\ell}x\right)$$

$$= \sum_n A_n \sin\left(\frac{n\pi}{\ell}x\right)$$

For ψ, we need a t-derivative:

$$u_t(x,t) = \sum_n \left[-A_n \left(\frac{n\pi}{\ell}c\right) \sin\left(\frac{n\pi}{\ell}ct\right) + B_n \left(\frac{n\pi}{\ell}c\right) \cos\left(\frac{n\pi}{\ell}ct\right)\right] \sin\left(\frac{n\pi}{\ell}x\right)$$

$$\psi(x) = u_t(x,0) = \sum_n \left[-A_n \frac{n\pi}{\ell}c \underbrace{\sin\left(\frac{n\pi}{\ell}c\cdot 0\right)}_{0} + B_n \frac{n\pi}{\ell}c \underbrace{\cos\left(\frac{n\pi}{\ell}c 0\right)}_{1}\right] \sin\left(\frac{n\pi}{\ell}x\right)$$

$$= \sum_n B_n \frac{n\pi}{\ell}c \sin\left(\frac{n\pi}{\ell}x\right)$$

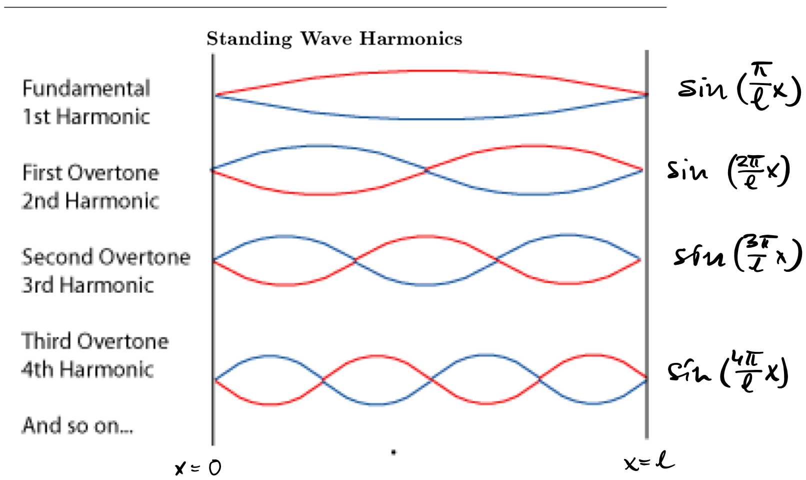
Harmonics: Let's go back and look at the basic solutions u_n, $n=1,2,3,\ldots$

$$u_1 = \underbrace{\left(A_1 \cos\left(\tfrac{\pi}{\ell}ct\right) + B_1 \sin\left(\tfrac{\pi}{\ell}ct\right)\right)} \sin\left(\tfrac{\pi}{\ell}x\right)$$

$$u_2 = \left(A_2 \cos\left(2\tfrac{\pi}{\ell}ct\right) + B_2 \sin\left(2\tfrac{\pi}{\ell}ct\right)\right) \sin\left(2\tfrac{\pi}{\ell}x\right)$$

$$u_3 = \underbrace{\left(A_3 \cos\left(3\tfrac{\pi}{\ell}ct\right) + B_3 \sin\left(3\tfrac{\pi}{\ell}ct\right)\right)}_{\text{temporal part}} \underbrace{\sin\left(3\tfrac{\pi}{\ell}x\right)}_{\text{spatial part}}$$

Here's how the first 4 of these look like:



They are called HARMONICS.

Each one has its own temporal behavior, called a frequency, given by the coefficient in the temporal part. The first frequencies are: $\frac{\pi}{\ell}c$, $\frac{2\pi}{\ell}c$, $\frac{3\pi}{\ell}c$, ……

Since $c = \sqrt{\frac{T}{\rho}}$ these are inherent properties of the string (depending on its tension, density, length...).