2.4 The Diffusion Equation On the Real Line

The textbook has a complicated derivation of a formula for the solution of the problem

$$\begin{cases} u_t(x,t) = k\, u_{xx}(x,t) & -\infty < x < \infty \quad t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

The formula turns out to be:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

(We only do some aspects. Only what we do here will be examinable; the discussion in the book will not feature in the exam.)

The Gaussian: The function

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}} \qquad (t > 0)$$

is called a Gaussian.

Properties:

a. $$\int_{-\infty}^{\infty} S(x,t)\,dx = 1 \qquad \forall\, t>0.$$

Proof: We do something that appears to complicate things: we include also the y-variable, and integrate in $\mathbb{R}^2$ instead of $\mathbb{R}$. This will end up making things easier.

$$\left(\int_{-\infty}^{\infty} S(x,t)\,dx\right)^2 = \left(\int_{-\infty}^{\infty} S(x,t)\,dx\right)\left(\int_{-\infty}^{\infty} S(y,t)\,dy\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x,t)\,S(y,t)\,dx\,dy$$
$$= \frac{1}{4\pi kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{4kt}}\,dx\,dy$$
$$= \frac{1}{4\pi kt}\int_{0}^{2\pi}\int_{0}^{\infty} e^{-\frac{r^2}{4kt}}\,r\,dr\,d\theta$$
$$= \frac{1}{4\pi kt}\cdot 2\pi\cdot(-2kt)\left[e^{-\frac{r^2}{4kt}}\right]_{r=0}^{\infty}$$
$$= 1.$$
$$\Longrightarrow \quad \int_{-\infty}^{\infty} S(x,t)\,dx = 1.$$

b. $$\forall\, x\neq 0 \qquad \lim_{t\downarrow 0} S(x,t) = 0.$$

Proof: Fix $x\neq 0$. Let $t=\frac{1}{4ks}$, $s=\frac{1}{4kt}$

$$\lim_{t\downarrow 0}\frac{1}{\sqrt{4\pi kt}}\,e^{-\frac{x^2}{4kt}} = \lim_{s\to\infty}\frac{1}{\sqrt{\pi}}\,\frac{\sqrt{s}}{e^{sx^2}} \underset{\text{l'Hôpital}}{=} \lim_{s\to\infty}\frac{1}{\sqrt{\pi}}\,\frac{1}{2\sqrt{s}\,x^2 e^{sx^2}} = 0$$

c. S satisfies the diffusion eq:

$$S_t = e^{-\frac{x^2}{4kt}} \left[-\frac{1}{2} \cdot \frac{1}{\sqrt{4\pi k}} \frac{1}{t^{3/2}} + \frac{1}{\sqrt{4\pi k t}} \frac{x^2}{4k} \frac{1}{t^2} \right]$$

$$S_x = \frac{1}{\sqrt{4\pi k t}} \left(\frac{-2x}{4kt} \right) e^{-\frac{x^2}{4kt}} = \frac{-2x}{\sqrt{\pi}\,(4kt)^{3/2}} e^{-\frac{x^2}{4kt}}$$

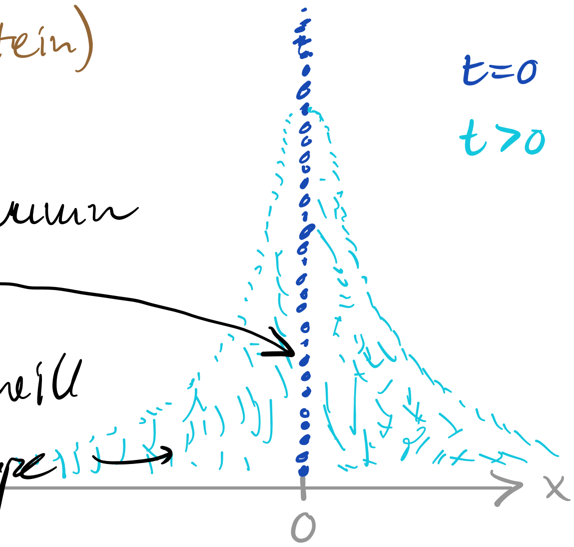
$$S_{xx} = e^{-\frac{x^2}{4kt}} \left[-\frac{2}{\sqrt{\pi}\,(4kt)^{3/2}} + \frac{4x^2}{\sqrt{\pi}\,(4kt)^{3/2}\, 4kt} \right]$$

Check that $S_t = k S_{xx}$.

Intuition: Here are two good ways to think about the meaning of $S(x,t)$.

Brownian motion: $S(x,t)$ is the probability to find a particle undergoing Brownian motion at the point x at time t if it started at $x=0$ at $t=0$. (this goes back to Einstein)

Sand: Imagine an infinite column of sand at $x=0$ at time $t=0$. Once we "turn on time" the sand will immediately fall. The resulting shape will be $S(x,t)$.



Conclusion: • A δ - "function" at time $t=0$ at $x=0$, will "become" $S(x,t)$ if it is the initial condition for $u_t = k u_{xx}$ on the real line.

- If it is at $x = x_0$ at $t=0$, then we get $S(x-x_0, t)$ instead.

- If we start with several δ-functions (i.e. several columns of sand) at $\{x_i\}_{i=0}^{N}$ then the result will be (from linearity)

$$\sum_{i=0}^{N} S(x - x_i, t).$$

- Leap of faith: If we start with $\phi(x)$, then it is like starting with infinitely many δ-functions, each at a different x and each weighted by $\phi(x)$, so we get

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y, t)\, \phi(y)\, dy \qquad t > 0.$$