2.3 The Diffusion Equation in an Interval

We saw that $\mathcal{L}_{wave}$ factors as

$$\mathcal{L}_{wave} = \mathcal{L}_c \mathcal{L}_{-c} = \mathcal{L}_{-c} \mathcal{L}_c .$$

With $\mathcal{L}_{dif} := \partial_t - k\,\partial_{xx}$ we can't do a similar trick, and that makes the diffusion equation a more complicated equation.

So we start by showing some important properties:

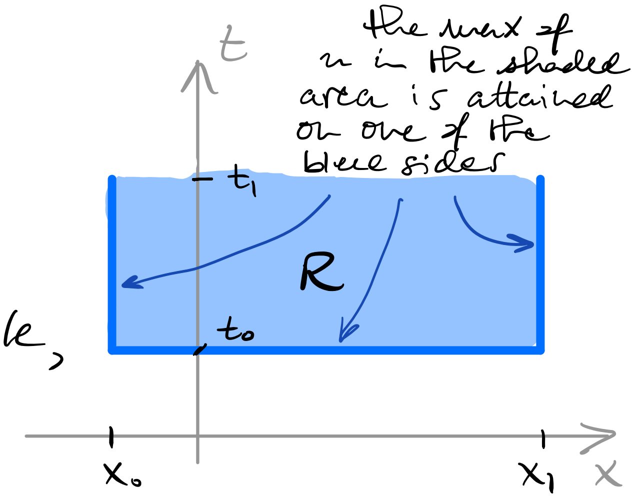
Theorem: (Maximum Principle)

Suppose that $u(x,t)$ is a solution of $u_t = k u_{xx}$.
Consider any rectangle

$$R := [x_0, x_1] \times [t_0, t_1]$$

in space-time.

Then $\max_R u(x,t) =$ the maximum of u in the rectangle, is attained either initially (on the side with $t = t_0$) or on one of the lateral sides ($x = x_0$ or $x = x_1$).



Proof: Denote $M = \max$ of u on: $\sqcup$

We need to show that M is also the max of u on: R

That is: we need to show that $u(x,t) \le M$ in R.

(Note that M is well-defined: $\sqcup$ is a closed and bounded set, and $u(x,t)$ is a continuous function, so it attains its max on $\sqcup$.)

Call this Γ

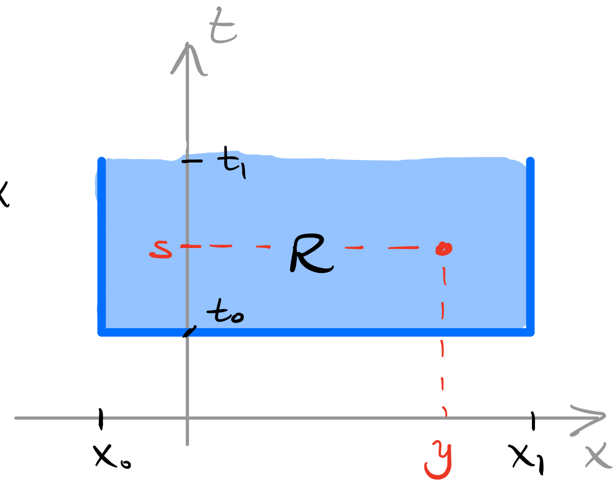
Let $\varepsilon > 0$. Define $v(x,t) = u(x,t) + \varepsilon x^2$.

We shall first prove that v attains its max on Γ, by contradiction.

$$\max_{\Gamma} v(x,t) = \max_{\Gamma} \left(u(x,t) + \varepsilon x^2\right)$$
$$\le \max_{\Gamma} u(x,t) + \max_{\Gamma} \varepsilon x^2$$
$$\le M + \varepsilon\left(|x_1| + |x_0|\right)^2$$

The function v satisfies:

$$v_t - k v_{xx} = u_t - k\left(u_{xx} + 2\varepsilon\right) =$$
$$= \underbrace{u_t - k u_{xx}}_{=0} - 2\varepsilon k = -2\varepsilon k < 0. \qquad (*)$$



Contradiction assumption:

Suppose that $v(x,t)$ attains its max at some point (y,s) inside R (in the interior of R). I.e.

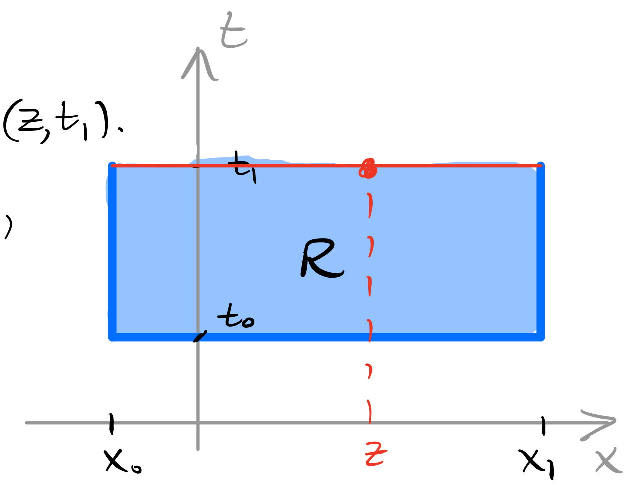
$$x_0 < y < x_1 \qquad t_0 < s < t_1.$$

$$\left.\begin{array}{llll} \text{Then} & V_t = 0 & \text{and} & V_x = 0 \\ \text{and} & V_{tt} \le 0 & \text{and} & V_{xx} \le 0 \end{array}\right\} \text{ at } (y,s)$$

So $V_t - kV_{xx} \ge 0$. But this contradicts $\circledast$.

Therefore $v(x,t)$ cannot attain its maximum <u>inside</u> the <u>interior</u> of R. We just need to rule out that the max is at the top edge of R.

By contradiction, assume that $v(x,t)$ attains a maximum on the top side at some point (z,t_1).



As before $V_x = 0$, $V_{xx} \le 0$ at (z,t_1).
But t derivatives are trickier, since we only know what happens for $t \le t_1$, but not $t > t_1$. The only thing we can say is that since $V(z,t_1)$ is a maximum, it must be $\ge v(z, t_1 - h)$ for $h>0$.
So we have:

$$V_t(z,t_1) = \lim_{h\to 0} \frac{v(z,t_1) - v(z,t_1-h)}{h} \ge 0$$

(where $h>0$)

So $V_t - kV_{xx} \ge 0$ again in contradiction to $\circledast$.

So the maximum of v cannot be inside the rectangle or on the top edge. Since it must be somewhere in the closed rectangle R (the maximum of a continuous function on a closed + bounded set is attained), the maximum must be attained on Γ.

Hence $v(x,t) \leq M + \varepsilon(|x_1| + |x_0|)^2$.

Now recall that $v(x,t) = u(x,t) + \varepsilon x^2$.

So

$$\begin{aligned} u(x,t) &= v(x,t) - \varepsilon x^2 \\ &\leq M + \varepsilon\left[(|x_1| + |x_0|)^2 - x^2\right] \qquad \forall\, t_0 \leq t \leq t_1,\ \forall\, x_0 \leq x \leq x_1 \end{aligned}$$

But this is true for any $\varepsilon > 0$. So it must hold that

$$u(x,t) \leq M \qquad \forall\, t_0 \leq t \leq t_1,\ \forall\, x_0 \leq x \leq x_1 .$$

$\blacksquare$

Important remark: We proved a theorem for the max of u. What makes the max of u more special than the min of u? NOTHING! There's no essential difference. Indeed, we can prove the same theorem for $\min_R u(x,t)$ by applying the theorem to $-u(x,t)$. The max of $-u$ is $-\min$ of u.

Conclusion: If $u(x,t)$ solves the diffusion equation, then both the minimum and the maximum of u in  are attained on .

Theorem: (Strong Maximum Principle)
The max (resp. min) of u lies strictly on  and not in the interior of  (with the exception of the case where u is constant throughout ).

THIS IS A DIFFICULT THEOREM WHICH WE ACCEPT WITHOUT PROOF