In this chapter we focus on the wave and diffusion equations on the whole real line $-\infty < x < +\infty$.
This is justified by "zooming in" to a very small subdomain of our domain: then the boundary seems so far away that it might as well be at $\pm\infty$.

2.1 The Wave Equation

We consider the equation

$$(*) \qquad u_{tt} = c^2 u_{xx} \quad \text{for } -\infty<x<+\infty, \quad t \geq 0.$$

First recall the following:

1) Solutions of $au_x + bu_y = 0$ have the form
$$u(x,y) = f(bx-ay).$$
Therefore solutions of $u_t + cu_x = 0$ have the form
$$u(x,t) = f(ct-x)$$

2) If $\mathcal{L}$ is linear, $\mathcal{L}w = g$ and $\mathcal{L}v = 0$ then $\mathcal{L}(w+v) = g$.
So to solve $\mathcal{L}w = g$ we need:
a. a particular solution w
b. any solution of the homogeneous problem.

Theorem: Any solution u of $(*)$ can be represented as
$$u(x,t) = f(x+ct) + g(x-ct)$$
for some functions $f, g \in C^2(\mathbb{R};\mathbb{R})$.

(this means that f, g are twice differentiable functions acting from $\mathbb{R}$ to $\mathbb{R}$)

Proof 1:

Let $\mathcal{L}_c = \partial_t + c\partial_x$, $\mathcal{L}_{-c} = \partial_t - c\partial_x$.

Then $\mathcal{L}_{wave} = \partial_{tt} - c^2\partial_{xx} =$

$$= (\partial_t - c\partial_x)(\partial_t + c\partial_x) = \mathcal{L}_{-c}\mathcal{L}_c$$

$$= (\partial_t + c\partial_x)(\partial_t - c\partial_x) = \mathcal{L}_c\mathcal{L}_{-c}$$

That is, the wave operator is a composition of left-right (or right-left) transport operators.

Consider $\mathcal{L}_{wave} = \mathcal{L}_{-c}\mathcal{L}_c$. Define $v = (\partial_t + c\partial_x)u = \mathcal{L}_c u$.

Then $\mathcal{L}_{wave}(u) = (\partial_t - c\partial_x)(\partial_x + c\partial_x)u = (\partial_t - c\partial_x)v = \mathcal{L}_{-c}v$

So $\mathcal{L}_{wave}(u) = 0 \iff \begin{cases} \mathcal{L}_c u = v & (1) \\ \mathcal{L}_{-c} v = 0 & (2) \end{cases}$

We know that (2) has the solution $v(x,t) = h(x+ct)$ where h is any function.

Considering (1), it now takes the form:

$$\mathcal{L}_c u = (\partial_t + c\partial_x)u = h(x+ct)$$

This is an inhomogeneous linear PDE (transport 1D).

The solution will be given by a particular solution + anything in the kernel of $\mathcal{L}_c$.

__Particular solution:__ We suspect that u is essentially the antiderivative of h. Let $H(w) = \int h(w)\, dw$, where $w = x+ct$.

$\partial_t H = H' \partial_t w = H' c$, $\quad c\partial_x H = cH' \partial_x w = cH'$

$\Rightarrow (\partial_t + c\partial_x) H = 2cH' = 2ch$

So, almost: it's not H, rather it is $\frac{H}{2c}$.

$$\mathcal{L}_c\left(\frac{H}{2c}\right) = h(x+ct).$$

__Homogeneous solution:__ We need a solution of $\mathcal{L}_c w = 0$. We already know that it has the form $w(x,t) = g(x-ct)$.

$$\Rightarrow \quad u(x,t) = f(x+ct) + g(x-ct).$$

(We can further verify this by also considering the other decomposition $\mathcal{L}_{wave} = \mathcal{L}_c \mathcal{L}_{-c}$).

$\blacksquare$

__Proof 2:__ Define $\quad \xi = x+ct \quad \eta = x-ct$.

$$\partial_x u(\xi,\eta) = \partial_\xi u \cdot \partial_x \xi + \partial_\eta u \cdot \partial_x \eta = (\partial_\xi + \partial_\eta) u(\xi,\eta)$$

$$\partial_t u(\xi,\eta) = \partial_\xi u \cdot \partial_t \xi + \partial_\eta u \cdot \partial_t \eta = c\partial_\xi u - c\partial_\eta u$$

$$= c(\partial_\xi - \partial_\eta) u$$

$$\mathcal{L}_c = \partial_t + c\partial_x = c(\partial_\xi - \partial_\eta) + c(\partial_\xi + \partial_\eta) = 2c\partial_\xi$$

$$\mathcal{L}_{-c} = \partial_t - c\partial_x = c(\partial_\xi - \partial_\eta) - c(\partial_\xi + \partial_\eta) = -2c\partial_\eta$$

Hence $\quad \mathcal{L}_{\text{wave}} u = \mathcal{L}_{-c} \mathcal{L}_{c} u$

$$= (-2c\,\partial_\eta)(2c\,\partial_\xi) u$$

$$= -4c^2 \partial_\eta \partial_\xi u$$

Since $c \neq 0$ we have $\quad \partial_\eta \partial_\xi u = 0$
which implies

$$u(x,t) = f(\xi) + g(\eta)$$

$$= f(x+ct) + g(x-ct).$$

(see Section 1.1 Example 3).

The initial value problem:

Now we consider:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty, \quad t \geq 0 \\ u(x,0) = \phi(x) & -\infty < x < \infty \\ u_t(x,0) = \psi(x) & -\infty < x < \infty \end{cases}$$

Theorem: (d'Alembert's formula)

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

Proof: We already know that u has the form $u(x,t) = f(x+ct) + g(x-ct)$.

Then: $u_t(x,t) = cf'(x+ct) - cg'(x-ct)$

Setting $t=0$ in these, we find:

$$\phi(x) = u(x,0) = f(x) + g(x)$$
$$\psi(x) = u_t(x,0) = cf'(x) - cg'(x)$$

Differentiating the first and dividing the second by c, we have:

$$\phi' = f' + g'$$
$$\frac{1}{c}\psi = f' - g'$$

$$\Longrightarrow \quad f' = \frac{1}{2}\left(\phi' + \frac{1}{c}\psi\right)$$
$$g' = \frac{1}{2}\left(\phi' - \frac{1}{c}\psi\right)$$

$$\Longrightarrow \quad f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi(r)\,dr + A$$
$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(r)\,dr + B$$

A, B being some constants. Since $\phi = f + g$ we know that $A + B = 0$.

Hence

$$u(x,t) = f(x+ct) + g(x-ct)$$
$$= \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi$$
$$= \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds$$

$\blacksquare$

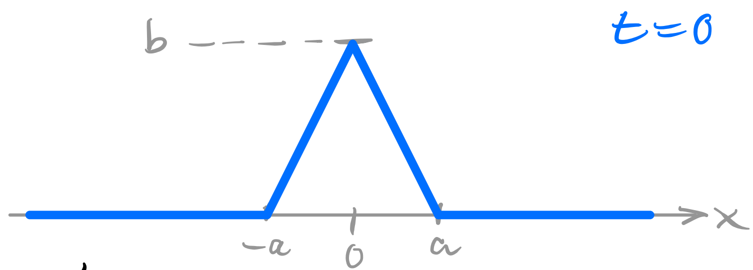
<u>Example</u>: Plucked string.

Consider a string with the initial conditions:

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases}$$

$$\psi(x) = 0$$

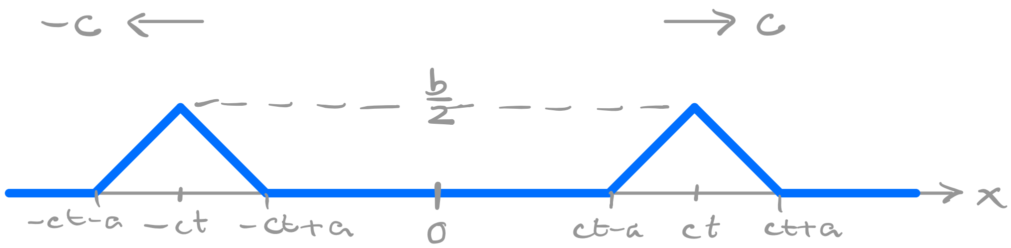
This means that we start at time $t=0$ with a static "plucked" string



We know that: $u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$

The wave will travel along the string with speed c to the left and to the right. So  will start flattening and eventually will result in 2 signals, one moving to the left and one to the right, each of them half the size of .

So, for very large times, we expect to see:



Based on this sketch, we expect the waves to separate when $ct - a = 0 \Rightarrow t = \frac{a}{c}$.

So we check for several times:

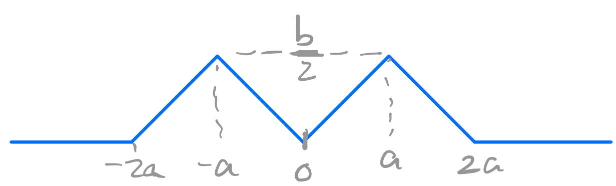
1. $t = \frac{a}{c}$: $u(x, \frac{a}{c}) = \frac{1}{2}[\phi(x+a) + \phi(x-a)]$

$$u(0, \tfrac{a}{c}) = \tfrac{1}{2}[\phi(a) + \phi(-a)] = 0$$

$$\begin{aligned} u(a, \tfrac{a}{c}) &= \tfrac{1}{2}[\phi(a+a) + \phi(a-a)] \\ &= \tfrac{1}{2}[0 + \phi(0)] \\ &= \tfrac{1}{2} b \end{aligned}$$

Similarly $u(-a, \frac{a}{c}) = \frac{1}{2} b$

Checking more points, we can find



2. check $t < \frac{a}{c}$

3. check $t > \frac{a}{c}$.