1.1 What are Partial Differential Equations?

$x, y, \ldots\ldots$ independent variables.
$u = u(x, y, \ldots)$.

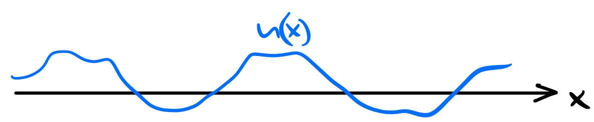
A partial differential equation (PDE) is an equation that relates $x, y, \ldots$, $u(x, y, \ldots)$ and partial derivatives of u w.r.t. $x, y, \ldots$

The order of a PDE is the highest derivative that appears.

1.3 Examples From the World Around Us

Ex 1: Transport in 1D

Consider some function $u(x)$



and suppose it moves at speed c to the right.
Then now $u = u(x,t)$ and

$$M = \int_0^b u(x,t)\,dx = \int_{0+ch}^{b+ch} u(x, t+h)\,dx$$

Differentiate w.r.t. b to get

$$\frac{\partial M}{\partial b} = u(b,t) = u(b+ch, t+h)$$

Differentiating this w.r.t. h we get

$$\frac{\partial}{\partial h}\left(\frac{\partial M}{\partial b}\right) = \frac{\partial}{\partial h}\left(u(b,t)\right) = 0$$

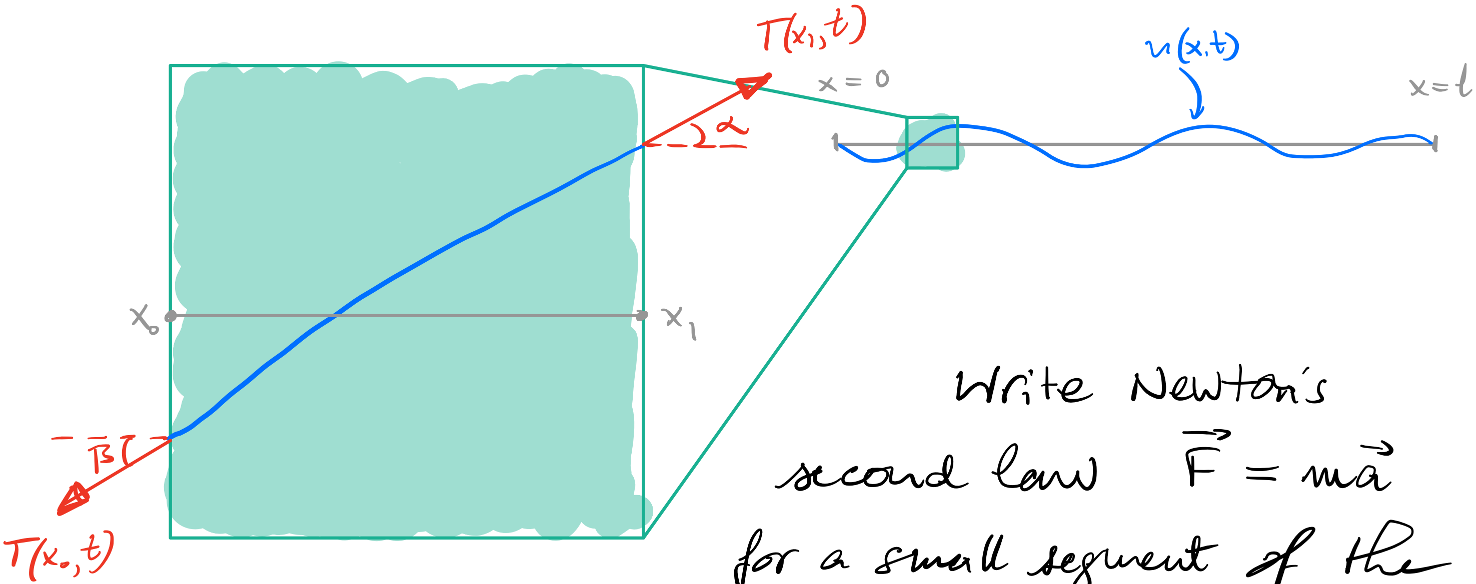
$$= \frac{\partial}{\partial h}\left(u(b+ch, t+h)\right) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial h}(b+ch) + \frac{\partial u}{\partial t}\frac{\partial t}{\partial h}(t+h)$$

$$= u_x(b+ch, t+h)\cdot c + u_t(b+ch, t+h)\cdot 1$$

Plugging in $h=0$ we have:

$$u_t(b,t) + c\,u_x(b,t) = 0$$

Ex 2: Vibrating string.



Write Newton's second law $\vec{F} = m\vec{a}$ for a small segment of the string, for horizontal & vertical parts separately.

Horizontal: $T(x_1,t)\cdot\cos\alpha - T(x_0,t)\cdot\cos\beta = 0$

$$\cos\alpha = \frac{1}{\sqrt{1+u_x^2(x_1,t)}} \approx \frac{1}{1+\frac{1}{2}u_x^2(x_1,t)} \approx 1$$

$$\cos\beta = \frac{1}{\sqrt{1+u_x^2(x_0,t)}} \approx \frac{1}{1+\frac{1}{2}u_x^2(x_0,t)} \approx 1$$

$$\Longrightarrow \qquad T(x_0,t) = T(x_1,t)$$

So T is independent of x. We assume that it is also independent of t.

<u>Vertical</u>: $\quad T(\sin\alpha - \sin\beta) = F = ma = \int_{x_0}^{x_1} \rho\, u_{tt}(x,t)\,dx$

$$\sin\alpha = \frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} \approx u_x(x_1,t)$$

$$\sin\beta \approx u_x(x_0,t)$$

$$\longrightarrow \qquad T\big(u_x(x_1,t) - u_x(x_0,t)\big) = \int_{x_0}^{x_1} \rho\, u_{tt}(x,t)\,dx$$

Replace $x_1 = x_0 + h$ and divide by h:

$$T \quad \frac{u_x(x_0+h,t) - u_x(x_0,t)}{h} = \frac{1}{h}\int_{x_0}^{x_0+h} \rho\, u_{tt}(x,t)\,dx$$

As $h \to 0$ we get:

$$c = \sqrt{\frac{T}{\rho}}$$

$$T u_{xx} = \rho u_{tt} \quad \longrightarrow \quad u_{tt} = c^2 u_{xx}$$

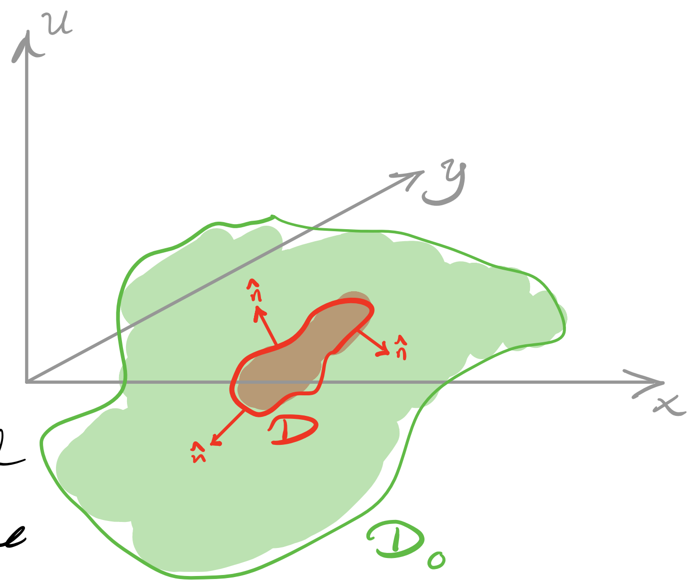
This is the wave equation, and c turns out to be the wave speed.

In 2D we get $\qquad u_{tt} = c^2(u_{xx} + u_{yy})$

3D $\qquad u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$

Theorem: ("Vanishing theorem")
Let open $D_0 \subset \mathbb{R}^n$, $n \geq 1$. Let $f : D_0 \longrightarrow \mathbb{R}$ be continuous satisfying $\iiint_D f(\vec{x})\, d\vec{x} = 0$ for any open $D \subset D_0$.
Then f is identically 0.

Ex 3: Vibrating drumhead.



Let $D_0 \subset \mathbb{R}^2$ be open.
Suppose that the boundary of D_0 is a frame holding a membrane (like a drum).
Let $u(x,y,t)$ denote the vertical displacement of the membrane at the point $(x,y) \in D_0$ at time $t \in \mathbb{R}$.

Let $D \subset D_0$ be any open subset (sometimes called domain). Let $\vec{T}(x,y,t)$ be the tension, and let $T(x,y,t) = \|\vec{T}(x,y,t)\|$ (magnitude). As before, the horizontal component guarantees that T doesn't depend on (x,y), and we assume again that it doesn't depend on t. Similar to the 1D case, the vertical contribution is given by

$$\int_{\partial D} T \frac{\partial u}{\partial n}\, ds = F = ma = \iint_D \rho\, u_{tt}\, dx\,dy$$

where $\hat{n}$ is the outward pointing unit normal vector to the boundary of D and $\frac{\partial u}{\partial n} = \hat{n} \cdot \nabla u$ is the directional derivative.

By Green's theorem (which is just the 2D version of the divergence theorem, aka Gauss' theorem)

$$\iint_D \nabla \cdot (T \nabla u)\, dx dy = \iint_D \rho u_{tt}\, dx dy$$

equivalently:
$$\iint_D \left(\nabla \cdot (T \nabla u) - \rho u_{tt} \right) dx dy = 0$$

Since D is arbitrary, the vanishing theorem implies that

$$\nabla \cdot (T \nabla u) - \rho u_{tt} = 0$$

and since T is constant we have

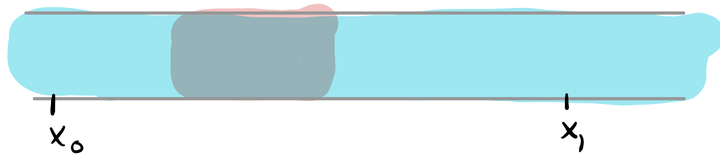
$$u_{tt} = c^2\, \nabla \cdot (\nabla u) = c^2 (u_{xx} + u_{yy})$$

where, as before, $c = \sqrt{\frac{T}{\rho}}$. In 3D

$$u_{tt} = c^2 (u_{xx} + u_{yy} + u_{zz})$$

We denote $\Delta u = u_{xx} + u_{yy} + u_{zz}$

Ex 4: Diffusion.



Fluid in a tube, and dye diffusing in it. The mass of the dye between x_0, x_1:

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx \qquad (u \text{ is the concentration})$$

$$\frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx$$

The mass can only change if dye flows in or out of this section of the tube. So

$$\frac{dM}{dt} = k u_x(x_1,t) - k u_x(x_0,t)$$

(k is a constant that takes care of units)

So we have

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = k u_x(x_1,t) - k u_x(x_0,t)$$

Again, letting $x_1 = x_0 + h$ and dividing by h, we find

$$u_t = k u_{xx}.$$

In 2D we get $\quad u_t = k(u_{xx} + u_{yy})$

3D $\quad u_t = k(u_{xx} + u_{yy} + u_{zz})$

Using the notation $\Delta = \frac{\partial^2}{\partial x^2} + \dots$ we find the general form:

$$\text{Wave:} \quad \partial_{tt} u = c^2 \Delta u$$

$$\text{Diffusion:} \quad \partial_t u = k \Delta u$$

Ex 5: Heat flow.

The diffusion eq. also describes heat flow. Read this.

Ex 6: Stationary waves and diffusions.

For both the wave eq. and the diffusion eq. if the solution doesn't actually depend on time then the time derivatives vanish and we're left with

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$$

This is called the Laplace eq. Solutions are harmonic functions.