1.1 What Are PDEs?

PDEs are equations that involve independent variables

$$x, y, z, t$$

as well as a function and its partial derivatives

$$u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$$

Typically, the equations we will encounter are of the form

$$F(x, y, z, t, u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}) = 0$$

<u>Definition</u>: The operator associated to a PDE is the mapping $\mathcal{L}$ that sends u to the expression appearing in the PDE involving u.

For example, for the wave eq. $u_{tt} = c^2 u_{xx}$, the associated operator is $\mathcal{L}u = u_{tt} - c^2 u_{xx}$.

<u>Definition:</u> An operator $\mathcal{L}$ is said to be linear if $\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v$ and $\mathcal{L}(au) = a\mathcal{L}u$ for any two functions u, v and constant a.

<u>Example:</u> The wave operator is linear. But the operator associated with the PDE $u_x + uu_y = 0$ isn't:

$$\mathcal{L}u = u_x + uu_y$$

$$\mathcal{L}(u+v) = (u+v)_x + (u+v)(u+v)_y$$

$$= u_x + uu_y + v_x + vv_y + uv_y + vu_y$$

<u>Definition:</u> Let $\mathcal{L}$ be a linear operator. The eq

$$\mathcal{L}u = 0$$

is called a homogeneous linear equation.

An equation of the form

$$\mathcal{L}u = g$$

where $g \neq 0$ is called an inhomogeneous linear equation.

<u>Example:</u> Let $\mathcal{L}u = u_t - k\Delta u$, the operator associated with the diffusion eq. Then

$\mathcal{L}u = 0$ is linear homogeneous

but $\mathcal{L}u = \sin t$ is linear inhomogeneous.

<u>Vector space structure:</u> Let $\mathcal{L}$ be a linear operator. If $\{u_i\}_{i=1}^n$ are all functions that solve $\mathcal{L}u=0$, then also $\mathcal{L}(\sum_{i=1}^n a_i u_i)=0$ for any constants a_i, $i=1,\ldots,n$.

Moreover, if v solves $\mathcal{L}v=g$, and u solves $\mathcal{L}u=0$,

then $u+v$ solves $\mathcal{L}(u+v)=g$.

So by solving the homogeneous eq. we can find infinitely more solutions to an inhomogeneous eq.