Section 5.2 Q8: (a) Prove that differentiation switches even to odd and odd to even.

(b) Prove the same for integration (ignoring the const. of int.).

(a) Let $f(x)$ be even: $f(x) = f(-x)$. From the definition of the derivative, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. At $-x$ we have:

$$\underbrace{\frac{f(-x+h)-f(-x)}{h}}_{\text{This tends to } f'(-x)} \underset{\text{since } f(x)=f(-x)}{=} \frac{f(x-h)-f(x)}{h} \underset{\text{replace } \eta=-h}{=} \frac{f(x+\eta)-f(x)}{-\eta} = \underbrace{-\frac{f(x+\eta)-f(x)}{\eta}}_{\text{This tends to } -f'(x)}$$

The other direction is similar.

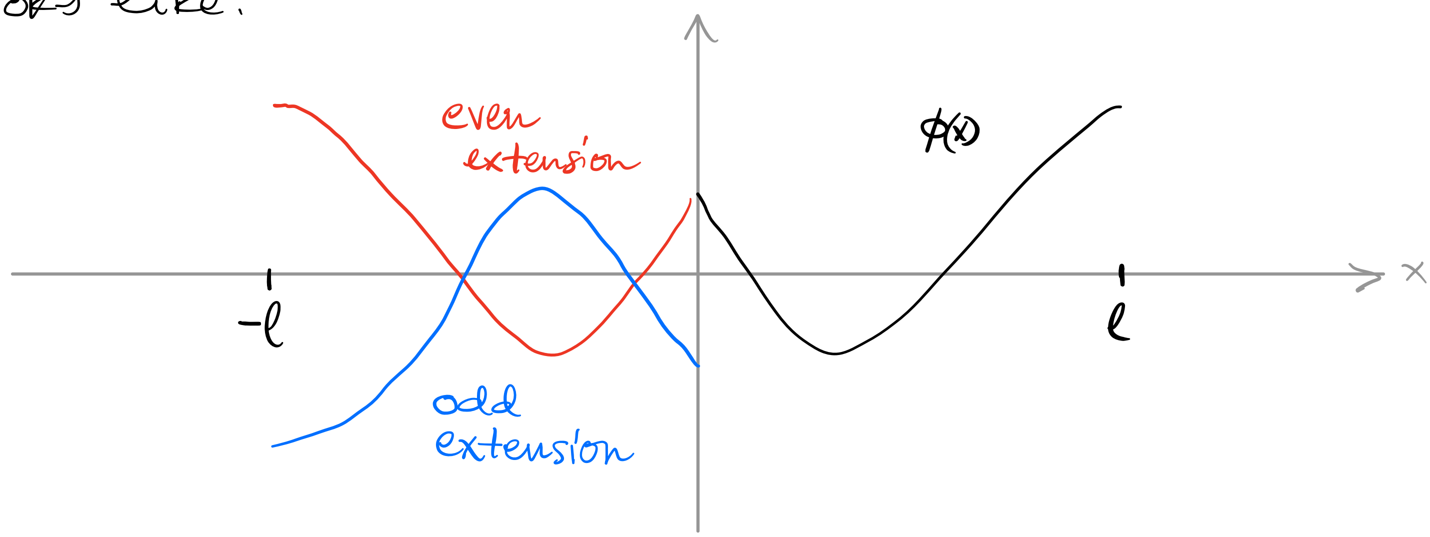
(b) Suppose $f(x) = f(-x)$. Define $g(x) = \int_0^x f(s)\,ds$. Then:

$$g(-x) = \int_0^{-x} f(s)\,ds = -\int_{-x}^0 f(s)\,ds = \int_x^0 f(-t)\,dt$$
$$= -\int_0^x f(-t)\,dt = -\int_0^x f(t)\,dt = -g(x)$$

The other direction is similar.

Section 5.2 Q10: (a) Let $\phi(x)$ be cont. on $(0,\ell)$. Under what conditions is its odd extension also a cont. function?
(b) Let $\phi(x)$ be a differentiable function on $(0,\ell)$. Under what conditions is its odd extension also a differentiable function?
(c) Same as (a) for even.
(d) Same as (b) for even.

First let's see graphically how an odd/even extension looks like:



(a) We have to require continuity at $x=0$. So we need the limits from the left and the right to agree:

$$\lim_{\varepsilon\downarrow 0} \phi(0+\varepsilon) \underset{\text{This is what we require}}{=} \lim_{\varepsilon\downarrow 0} \phi(0-\varepsilon) \underset{\text{This is from oddness}}{=} \lim_{\varepsilon\downarrow 0} -\phi(0+\varepsilon)$$

So we require that $\underbrace{\phi(0+)}_{\text{limit from right}} = -\underbrace{\phi(0-)}_{\text{limit from left}}$

$$\longrightarrow \phi(0+) + \phi(0-) = 0.$$

Since we want ϕ to be continuous, $\phi(0-)=\phi(0+)=\phi(0)$.
So we have: $2\phi(0)=0 \longrightarrow \phi(0)=0$.

(b) We need for ϕ to be continuous (i.e. $\phi(0)=0$ from (a)) but we also need the derivatives from the right and left at 0 to equal one another:

here we use oddness

$$\lim_{\varepsilon \searrow 0} \frac{\phi(0+\varepsilon) - \underbrace{\phi(0)}_{=0}}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\underbrace{\phi(0)}_{=0} - \phi(0-\varepsilon)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\phi(\varepsilon)}{\varepsilon}$$

$$\Longrightarrow \lim_{\varepsilon \searrow 0} \frac{\phi(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\phi(\varepsilon)}{\varepsilon}$$ which is a tautology. So there's no condition to impose.

(c) We require: $\phi(0+) = \phi(0-)$

This will always hold, since ϕ is assumed to be continuous, so that $\phi(0+) = \phi(0-) = \phi(0)$.

So there's no condition to impose.

(d) We require: $$\lim_{\varepsilon \searrow 0} \frac{\phi(0+\varepsilon) - \phi(0)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\phi(0) - \underbrace{\phi(0-\varepsilon)}_{=\phi(\varepsilon) \text{ by evenness}}}{\varepsilon}$$

$$\Longrightarrow 2 \lim_{\varepsilon \searrow 0} \frac{\phi(\varepsilon) - \phi(0)}{\varepsilon} = 0$$

$$\Longrightarrow \lim_{\varepsilon \searrow 0} \frac{\phi(\varepsilon) - \phi(0)}{\varepsilon} = 0$$

$\Longrightarrow$ The right derivative of $\phi(x)$ at $x=0$ is 0.

Section 5.2 Q17: Show that a complex-valued function $f(x)$ is real-valued if and only if its complex Fourier coefficients satisfy $c_n = c_{-n}^*$.

Write $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ (for simplicity take $l=\pi$).

in this step we swap $n \leftrightarrow -n$

Assume that $f(x)$ is real-valued. So $f(x) = \overline{f(x)}$.

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x) = \overline{f(x)} = \overline{\sum_{n\in\mathbb{Z}} c_n e^{inx}} = \sum_{n\in\mathbb{Z}} c_n^* \overline{e^{inx}} = \sum_{n\in\mathbb{Z}} c_n^* e^{-inx} = \sum_{n\in\mathbb{Z}} c_{-n}^* e^{inx}$$

So we find that $\sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\in\mathbb{Z}} c_{-n}^* e^{inx}$. Since $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal basis, any expansion of a function in these basis functions is unique, i.e. the coefficients must be the same:

$$c_n = c_{-n}^*.$$

Conversely, assume that $c_n = c_{-n}^*$.

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\in\mathbb{Z}} c_{-n}^* e^{inx} = \sum_{n\in\mathbb{Z}} c_n^* e^{-inx} = \sum_{n\in\mathbb{Z}} c_n^* \overline{e^{inx}} =$$

$$= \overline{\sum_{n\in\mathbb{Z}} c_n e^{inx}} = \overline{f(x)}.$$

Section 5.3 Q1: (a) Find the real vectors that are orthogonal to the given vectors $(1,1,1)$ and $(1,-1,0)$.

(b) Choosing an answer to (a), expand the vector $(2,-3,5)$ as a linear combination of these three mutually orthogonal vectors.

(a) We look for a vector (x,y,z) that is orthogonal to $(1,1,1)$ and $(1,-1,0)$.

1. $(x,y,z)\cdot(1,1,1) = 0 \longrightarrow x+y+z=0 \Rightarrow z=-(x+y)$

2. $(x,y,z)\cdot(1,-1,0) = 0 \longrightarrow x-y=0 \Rightarrow x=y$

Choose $x=1 \Rightarrow y=1 \Rightarrow z=-2$: multiples of $(1,1,-2)$ are orthogonal to both given vectors.

(b) Denote $v_1=(1,1,1)$ $v_2=(1,-1,0)$ $v_3=(1,1,-2)$

$v=(2,-3,5)$. Writing $v=\sum_{i=1}^{3} a_i v_i$, we seek a_i. The formula for a_i is: $a_i = \frac{1}{\|v_i\|^2}(v,v_i)$.

$\|v_1\|^2 = 1^2+1^2+1^2=3$. $(v,v_1)=2-3+5=3 \Rightarrow a_1=\frac{3}{3}=1$.

$\|v_2\|^2 = 1^2+(-1)^2+0^2=2$. $(v,v_2)=2+3=5 \Rightarrow a_2=\frac{5}{2}$.

$\|v_3\|^2 = 1^2+1^2+(-2)^2=6$. $(v,v_3)=2-3-10=-11 \Rightarrow a_3=-\frac{11}{6}$

$$\Rightarrow \quad v = 1\cdot v_1 + \frac{5}{2}v_2 - \frac{11}{6}v_3.$$

Section 5.3 Q2: (a) On the interval $[-1,1]$, show that the function $f(x) = x$ is orthogonal to the constant functions.
(b) Find a quadratic polynomial that is orthogonal to both 1 and x.
(c) Find a cubic polynomial that is orthogonal to all quadratics.

(a) Let $g(x) = c$ where c is a constant. Then:

$$(f,g) = \int_{-1}^{1} x \cdot c \, dx = c\left[\frac{x^2}{2}\right]_{x=-1}^{1} = c\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

$\Rightarrow$ by definition f and g are orthogonal.

(b) Let $h(x) = Ax^2 + Bx + C$. We first compute (h,f) and (h,g).

$$(h,f) = \int_{-1}^{1} (Ax^2 + Bx + C)\, x \, dx = \int_{-1}^{1} (Ax^3 + Bx^2 + Cx)\, dx$$

$$= \left[\frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2\right]_{x=-1}^{1} = \frac{A}{4} + \frac{B}{3} + \frac{C}{2} - \left[\frac{A}{4} - \frac{B}{3} + \frac{C}{2}\right] = \frac{2}{3}B$$

$$(h,g) = \int_{-1}^{1} [Ax^2 + Bx + C]\, 1 \, dx = \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_{x=-1}^{1}$$

$$= \frac{2}{3}A + 2C$$

WLOG we take $g(x) = 1$ rather than $g(x) = c$

Both of these will have to vanish, so we can choose $B = 0$, $A = -3C$. We have the freedom to choose C as we wish (just not 0), so we take $C = -1$ which gives us: $h(x) = 3x^2 - 1$. This polynomial is orthogonal to both 1 and x.

(c) Let $k(x) = Ax^3 + Bx^2 + Cx + D$. We first compute:

$$(k,f) = \int_{-1}^{1} (Ax^3 + Bx^2 + Cx + D)\, x \, dx = \left[\frac{A}{5}x^5 + \frac{B}{4}x^4 + \frac{C}{3}x^3 + \frac{D}{2}x^2\right]_{x=-1}^{1}$$

$$= \frac{2A}{5} + \frac{2C}{3}$$

$$(k,g) = \int_{-1}^{1} (Ax^3 + Bx^2 + Cx + D)\, dx = \frac{2B}{3} + 2D$$

$$(k,h) = \int_{-1}^{1} (Ax^3 + Bx^2 + Cx + D)(3x^2 - 1)\, dx =$$
$$= \int_{-1}^{1} (3Ax^5 + 3Bx^4 + (3C - A)x^3 + (3D - B)x^2 - Cx - D)\, dx$$
$$= \frac{6}{5}B + \frac{2(3D-B)}{3} - 2D = \left(\frac{6}{5} - \frac{2}{3}\right)B + \cancel{2D} - \cancel{2D}$$

We need all these inner products to be 0. From $(k,h)=0$ we find that $B=0$. Combined with $(k,g)=0$ this leads to $D=0$.

So we are left with $\frac{2A}{5} = -\frac{2C}{3} \implies A = -\frac{5C}{3}$. Choosing $C = -3$ we have: $A = 5$. We conclude:

$$k(x) = 5x^3 - 3x.$$

<u>Section 5.3 Q4:</u> Consider the problem

$$\begin{cases} u_t = k u_{xx} & 0<x<l \quad t>0 \\ u(0,t) = U & t>0 \\ u_x(l,t) = 0 & t>0 \\ u(x,0) = 0 & 0<x<l \end{cases}$$

(a) Find the solution in series form.

(b) Show that the series converges for $t>0$.

(c) Given $\varepsilon > 0$, estimate how long a time is required for $u(l,t)$ to be approximated by U to within ε error.

(a) Define $v(x,t) = u(x,t) - U$.

Then v satisfies the problem:

$$\begin{cases} v_t = k v_{xx} & 0<x<l \quad t>0 \\ v(0,t) = 0 & t>0 \\ v_x(l,t) = 0 & t>0 \\ v(x,0) = -U & 0<x<l \end{cases}$$

We have seen this problem already in Section 4.2 Q1. The solution was

$$v(x,t) = \sum_{n=0}^{\infty} C_n e^{-k\frac{\pi^2}{l^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{l}\left(n+\frac{1}{2}\right)x\right).$$

notice that the sum starts from $n=0$

We need to use the initial condition $v(x,0) = -U$:

$$-U = v(x,0) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)$$

$$\longrightarrow \quad C_n = \frac{2}{\ell}\int_0^{\ell} (-U) \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right) dx = \frac{2U}{(n+\frac{1}{2})\pi} \underbrace{\left[\cos\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)\right]_{x=0}^{\ell}}_{=0-1=-1}$$

$$= -\frac{4U}{(2n+1)\pi}$$

$$\longrightarrow \quad v(x,t) = -\sum_{n=0}^{\infty} \frac{4U}{(2n+1)\pi} e^{-k\frac{\pi^2}{\ell^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right).$$

$$\longrightarrow \quad u(x,t) = U - \sum_{n=0}^{\infty} \frac{4U}{(2n+1)\pi} e^{-k\frac{\pi^2}{\ell^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)$$

(b) The terms in this series are:

$$a_n = \frac{4U}{(2n+1)\pi} e^{-k\frac{\pi^2}{\ell^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)$$

$$= \frac{A}{2n+1} e^{-[Bn^2+Cn+D]} \sin\left(E\left(n+\frac{1}{2}\right)\right)$$

$$\begin{cases} A = \frac{4U}{\pi} \\ B = k\frac{\pi^2}{\ell^2}t \\ C = k\frac{\pi^2}{\ell^2}t \\ D = \frac{1}{4}k\frac{\pi^2}{\ell^2}t \\ E = \frac{\pi}{\ell}x \end{cases}$$

We bound $|a_n| \leq b_n := \frac{\tilde{A}}{2n+1} e^{-Bn^2} e^{-Cn}$ $\qquad \tilde{A} = A \cdot e^{-D}$

We now show that $\sum_{n=0}^{\infty} b_n$ converges, thereby implying that $\sum_{n=0}^{\infty} a_n$ converges absolutely. To show that $\sum_{n=0}^{\infty} b_n$ converges we use the ratio test:

$$\frac{b_{n+1}}{b_n} = \frac{2n+1}{2n+3} e^{-B[(n+1)^2-n^2]} e^{-C((n+1)-n)} = \frac{2+\frac{1}{n}}{2+\frac{3}{n}} e^{-2Bn} e^{-B-C}$$

$$\longrightarrow \quad \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 0.$$ Hence $\sum_{n=0}^{\infty} b_n < +\infty$, so that $\sum_{n=0}^{\infty} a_n < +\infty$.

(c) The expression for $u(\ell,t)$ is:

$$u(\ell,t) = U - \sum_{n=0}^{\infty} \underbrace{\frac{4U}{(2n+1)\pi} e^{-k\frac{\pi^2}{\ell^2}(n+\frac{1}{2})^2 t}}_{\text{call this part } d_n} \overbrace{\sin(\pi(n+\tfrac{1}{2}))}^{=(-1)^n}$$

$$\Rightarrow u(\ell,t) = U - d_0 + d_1 - d_2 + d_3 - + \cdots$$

where we already know that $\sum_{n=0}^{\infty} |d_n| < +\infty$ from (b).

Since this is an alternating series, $|u(\ell,t) - U| < |d_0| = \frac{4|U|}{\pi} e^{-kt\frac{\pi^2}{4\ell^2}}$

To get $|d_0| < \varepsilon$ we need $\frac{4|U|}{\pi} e^{-kt\frac{\pi^2}{4\ell^2}} < \varepsilon$

$$\rightarrow \quad e^{-kt\frac{\pi^2}{4\ell^2}} < \frac{\varepsilon\pi}{4|U|} \quad \rightarrow \quad -kt\frac{\pi^2}{4\ell^2} < \ln\left(\frac{\varepsilon\pi}{4|U|}\right)$$

$$\Longrightarrow \quad t > \frac{4\ell^2}{k\pi^2} \ln\left(\frac{4|U|}{\varepsilon\pi}\right).$$

Section 5.3 Q6: Find the complex eigenvalues of $\frac{d}{dx}$ subject to the BCs $X(0) = X(1)$. Are the eigenfunctions orthogonal on $(0,1)$?

We need to solve $X'(x) = \lambda X(x)$, $X(0) = X(1)$.

The eq. has solution $X(x) = e^{\lambda x}$. Requiring $X(0) = X(1)$ leads to $1 = e^{\lambda}$ $\rightarrow$ $\lambda_n = 2n\pi i$, $n \in \mathbb{Z}$

The eigenfunctions are $X_n(x) = e^{2n\pi i x}$, which satisfy:

$$(X_n, X_m) = \int_0^1 e^{2n\pi i x} e^{-2m\pi i x}\, dx = \int_0^1 e^{2\pi i x(n-m)}\, dx =$$

$$= \frac{1}{2\pi i(n-m)} \left[e^{2\pi i x(n-m)}\right]_{x=0}^{1} = 0 \quad \text{whenever } n \neq m.$$

So for $n \neq m$, X_n and X_m are orthogonal.

Section 5.3 Q12: Prove Green's First Identity:

$$\int_a^b f''(x)\, g(x)\, dx = -\int_a^b f'(x)\, g'(x)\, dx + f'g \Big|_{x=a}^{b} .$$

This is just integration by parts with: $u(x) = g(x)$, $v'(x) = f''(x)$.