Section 4.1 Q1: (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by an octave when the string is clamped exactly at its midpoint.
(b) Explain why the note rises when the string is tightened.

Suppose that our violin has a string of length ℓ, density ρ and tension T. Then it behaves according to the wave eq. with Dirichlet BCs (boundary conditions):

$$\left(c = \sqrt{\tfrac{T}{\rho}}\right) \qquad \begin{cases} u_{tt} = c^2 u_{xx} & 0<x<\ell \quad t>0 \\ u(0,t) = u(\ell,t) = 0 & t \geq 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) & 0 \leq x \leq \ell \end{cases}$$

We have seen that the solution to this problem has the form:

$$u(x,t) = \sum_n \left[A_n \cos\left(\tfrac{n\pi}{\ell} ct\right) + B_n \sin\left(\tfrac{n\pi}{\ell} ct\right)\right] \sin\left(\tfrac{n\pi}{\ell} x\right)$$

The frequencies of the harmonics are given by

$$\frac{n\pi}{\ell} c = \frac{n\pi}{\ell}\sqrt{\frac{T}{\rho}}$$

(a) If the string is clamped at the midpoint, it would be equivalent to solving the above problem with ℓ replaced by $\frac{\ell}{2}$ everywhere.

So the frequencies become $\frac{n\pi}{\ell/2} c = 2\,\frac{n\pi}{\ell} c$

That is: the frequencies of all harmonics double.

(frequency doubling means an octave higher)

(b) If the string is tightened, say instead of T we have $T^* > T$ then the frequencies become

$$\frac{n\pi}{\ell}\sqrt{\frac{T^*}{\rho}} > \frac{n\pi}{\ell}\sqrt{\frac{T}{\rho}}.$$

That is, the frequencies of all harmonics grow, i.e. the notes rise.

Section 4.1 Q2: Consider a metal rod $(0<x<\ell)$, insulated along its sides but not at its ends, which is initially at temperature 1. Suddenly, both ends are plunged into a bath of temp. 0. Write the PDE + BC + IC. Write the formula for the temp. $u(x,t)$ at later times. You can assume the infinite series expansion: $1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\sin\left(\frac{\pi}{\ell}(2n-1)x\right)\right]$.

The equation is:

$$\begin{cases} u_t(x,t) = k u_{xx}(x,t) & 0<x<\ell \quad t>0 \quad \text{(PDE)} \\ u(0,t) = u(\ell,t) = 0 & t>0 \quad \text{(BC)} \\ u(x,0) = 1 & 0\le x\le \ell \quad \text{(IC)} \end{cases}$$

The solution is given by (as we've seen in class)

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\left(\frac{n\pi}{\ell}x\right)$$

Now we need to use the IC to determine the A_n's:

$$1 = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right)$$

We use the fact that: $1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\sin\left(\frac{\pi}{\ell}(2n-1)x\right)\right]$

To get: $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)}\sin\left(\frac{\pi}{\ell}(2n-1)x\right)$

This can only be true if all the even A_n's are 0, and for $m = 2n-1$, $A_m = \frac{4}{\pi m}$.

So we have: $$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} e^{-\left(\frac{(2n-1)\pi}{\ell}\right)^2 kt} \sin\left(\frac{(2n-1)\pi}{\ell} x\right)$$

Alternatively, this can be written as:

$$u(x,t) = \sum_{m \text{ odd}} \frac{4}{m\pi} e^{-\left(\frac{m\pi}{\ell}\right)^2 kt} \sin\left(\frac{m\pi}{\ell} x\right)$$

Section 4.1 Q3: A quantum mechanical particle on the line with an infinite potential outside the interval $(0,\ell)$ is given by Schrödinger's eq. $u_t = i u_{xx}$ on $(0,\ell)$ with Dirichlet conditions at the ends. Separate the variables and represent the solution as a series.

We face the problem: $$\begin{cases} u_t(x,t) = i u_{xx}(x,t) & 0<x<\ell \quad t>0 \\ u(0,t) = u(\ell,t) = 0 \end{cases}$$

Separate variables: $u(x,t) = X(x)\,T(t)$ and plug into eq:

$$X(t)\,T'(t) = i\,X''(x)\,T(t)$$

Divide by iXT: $$\frac{1}{i}\frac{T'}{T} = \frac{X''}{X}$$

function of t   function of x

These can equal only if they are both constant. Call it $-\lambda$.

Then we have: $-\frac{1}{i}\frac{T'}{T} = -\frac{X''}{X} = \lambda = \beta^2$

X part: As in class, we get

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The Dirichlet BCs, just like in class, require

$$A = 0$$

and $\beta_n = \frac{n\pi}{\ell}$

So we get the eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right)$
and eigenvalues $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$.

T part: We now have: $T'(t) = -i\lambda T(t)$

$$\Longrightarrow \quad T_n(t) = e^{-i\lambda_n t}$$

$$\Longrightarrow \quad u_n(x,t) = A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

$$\Longrightarrow \quad u(x,t) = \sum_n A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

Section 4.2 Q1 :

Solve the diffusion problem

$$\begin{cases} u_t(x,t) = k\, u_{xx}(x,t) & 0<x<\ell \quad t>0 \\ u(0,t) = u_x(\ell,t) = 0 \end{cases}$$

Separation of variables $u(x,t) = X(x)\,T(t)$ leads to the expression we've seen for the diffusion eq:

$$-\frac{T'}{kT} = -\frac{X''}{X} = \lambda \quad (= \beta^2)$$

As before, the X part gives solutions of the form:

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$
$$X'(x) = -A\beta\sin(\beta x) + B\beta\cos(\beta x)$$

BCs are: $X(0) = X'(\ell) = 0$. Plug this in:

$$0 = X(0) = A\underbrace{\cos 0}_{1} + B\underbrace{\sin 0}_{0} = A \implies A = 0.$$
$$0 = X'(\ell) = B\beta\cos(\beta\ell) \implies \cos(\beta\ell) = 0$$

$$\implies \beta\ell = \frac{\pi}{2} + n\pi = \pi\left(n+\tfrac{1}{2}\right)$$

$$\implies \beta_n = \frac{\pi}{\ell}\left(n+\tfrac{1}{2}\right) \qquad \lambda_n = \frac{\pi^2}{\ell^2}\left(n+\tfrac{1}{2}\right)^2$$

EIGENVALUES

EIGENFUNCTIONS: $X_n(x) = \sin\left(\frac{\pi}{\ell}\left(n+\tfrac{1}{2}\right)x\right)$

We should check if 0 can be an eigenvalue:

If it is, then the eq. for X gives us $X''=0$ which becomes $X(x) = D + Cx$.

Check the BCs:

$$\left.\begin{aligned} 0 &= X(0) = D \\ 0 &= X'(\ell) = C \end{aligned}\right\} \Longrightarrow \text{so both } C=D=0$$

which means that the solution is trivial: $X(x)=0$

$\Longrightarrow$ 0 is <u>not</u> an eigenvalue.

So we can proceed with the temporal part as in class; it gives: $T_n(t) = e^{-k\lambda_n t}$

In conclusion, we find solutions of the form

$$u_n(x,t) = C_n e^{-k\frac{\pi^2}{\ell^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)$$

and the general solution has the form:

$$u(x,t) = \sum_n C_n e^{-k\frac{\pi^2}{\ell^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right)x\right)$$

Section 4.2 Q2: Consider the eq. $\begin{cases} u_{tt} = c^2 u_{xx} & 0<x<l,\ t>0 \\ u_x(0,t) = u(l,t) = 0 \end{cases}$

(a) Show that the eigenfunctions are

$$\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi}{l}x\right)$$

(b) Write the series expansion solution.

As before, separation of variables gives

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda \qquad (=\beta^2)$$

$$\longrightarrow \quad X(x) = A\cos\beta x + B\sin\beta x$$
$$X'(x) = -A\beta\sin\beta x + B\beta\cos\beta x$$

$u_x(0,t)=0 \implies 0 = X'(0) = -A\beta\underbrace{\sin 0}_{0} + B\beta\underbrace{\cos 0}_{1} = B\beta$

$\implies$ since $\beta \neq 0$ we conclude $B=0$.

$u(l,t)=0 \implies 0 = X(l) = A\cos\beta l \longrightarrow \cos\beta l = 0$

$\implies \beta l = \frac{\pi}{2} + n\pi = \pi\left(n+\frac{1}{2}\right)$

$\longrightarrow \beta_n = \frac{\pi}{l}\left(n+\frac{1}{2}\right) \qquad \lambda_n = \frac{\pi^2}{l^2}\left(n+\frac{1}{2}\right)^2$

$$\longrightarrow \quad X_n(x) = \cos\left(\frac{\pi}{l}\left(n+\tfrac{1}{2}\right)x\right)$$

Is 0 an eigenvalue? If so, then $X(x) = D + Cx$.

$$\left.\begin{aligned} 0 &= X'(0) = C \\ 0 &= X(l) = D \end{aligned}\right\} \quad C = D = 0 \implies X(x) \text{ is trivial}$$

$\implies$ 0 not an eigenvalue.

The T part is as before (in class) for the wave eq:

$$T_n(t) = A_n \cos(\beta_n c t) + B_n \sin(\beta_n c t)$$

So we get:

$$u_n(x,t) = \left[A_n \cos\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) ct\right) + B_n \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) ct\right)\right] \cos\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) x\right)$$

and

$$u(x,t) = \sum_n \left[A_n \cos\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) ct\right) + B_n \sin\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) ct\right)\right] \cos\left(\frac{\pi}{\ell}\left(n+\frac{1}{2}\right) x\right)$$

Section 4.2 Q3: Solve the Schrödinger eq.

$$\begin{cases} u_t = iku_{xx} & 0<x<l \quad t>0 \\ u_x(0,t) = u(l,t) = 0 & t>0 \end{cases}$$

where $k \in \mathbb{R}$.

Separation of variables gives:

$$-\frac{T'}{ikT} = -\frac{X''}{X} = \lambda \qquad (=\beta^2)$$

X part:

$$X(x) = C\cos\beta x + D\sin\beta x$$
$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$$

$u_x(0,t)=0 \Rightarrow 0 = X'(0) = -C\beta \underbrace{\sin 0}_{0} + D\beta \underbrace{\cos 0}_{1} = D\beta \Rightarrow D=0$

$u(l,t)=0 \Rightarrow 0 = X(l) = C\cos\beta l \Rightarrow \cos\beta l = 0$

$$\Rightarrow \beta l = \frac{\pi}{2} + n\pi = \pi(n+\tfrac{1}{2})$$

$$\Rightarrow \beta_n = \frac{\pi}{l}(n+\tfrac{1}{2}) \qquad \lambda_n = \left(\frac{\pi}{l}\right)^2 (n+\tfrac{1}{2})^2$$

$$X_n(x) = \cos\left(\frac{\pi}{l}(n+\tfrac{1}{2})x\right)$$

T part:

$$T' = -ik\lambda T$$
$$\Rightarrow T_n(t) = e^{-ik\lambda_n t}$$

$$\Rightarrow u_n(x,t) = A_n e^{-ik\left(\frac{\pi}{l}\right)^2 (n+\frac{1}{2})^2 t} \cos\left(\frac{\pi}{l}(n+\tfrac{1}{2})x\right)$$

$$\Rightarrow u(x,t) = \sum_n A_n e^{-ik\left(\frac{\pi}{l}\right)^2 (n+\frac{1}{2})^2 t} \cos\left(\frac{\pi}{l}(n+\tfrac{1}{2})x\right)$$