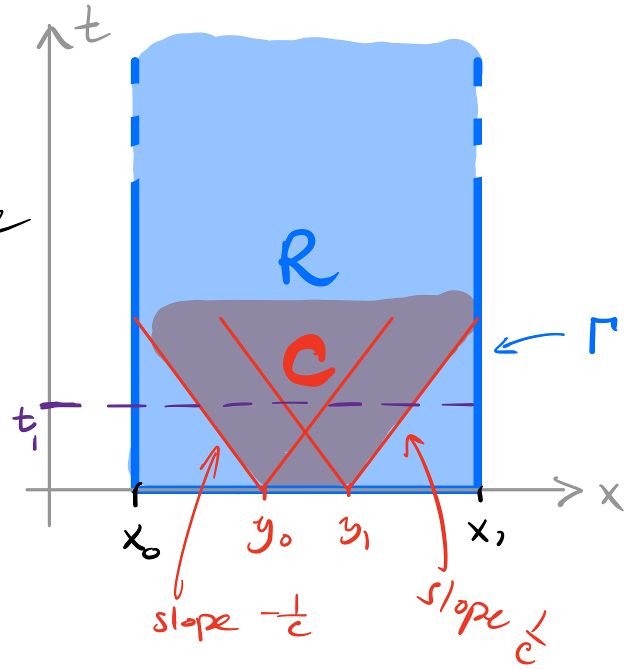
Section 2.5 Q1: Show that there is no maximum principle for the wave equation.



As usual, let R be a rectangle in the x−t plane and let Γ be the union of its bottom, right and left sides.

We consider the problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty<x<\infty \quad t>0 \\ u(x,0)=\phi(x) \quad u_t(x,0)=\psi(x) & -\infty<x<\infty \end{cases}$$

We know the solution to be given by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

Idea of the proof:

- Take initial conditions $\phi(x)=0$ and $\psi(x)$ = nonzero in a "small" region. Call this "small" region (y_0, y_1) (see sketch).
- Draw the domains of influence of (y_0,t), (y_1,t). The domain of influence of the interval between y_0 and y_1 is shaded red. Call it C.
- Take a small time t_1, smaller than when

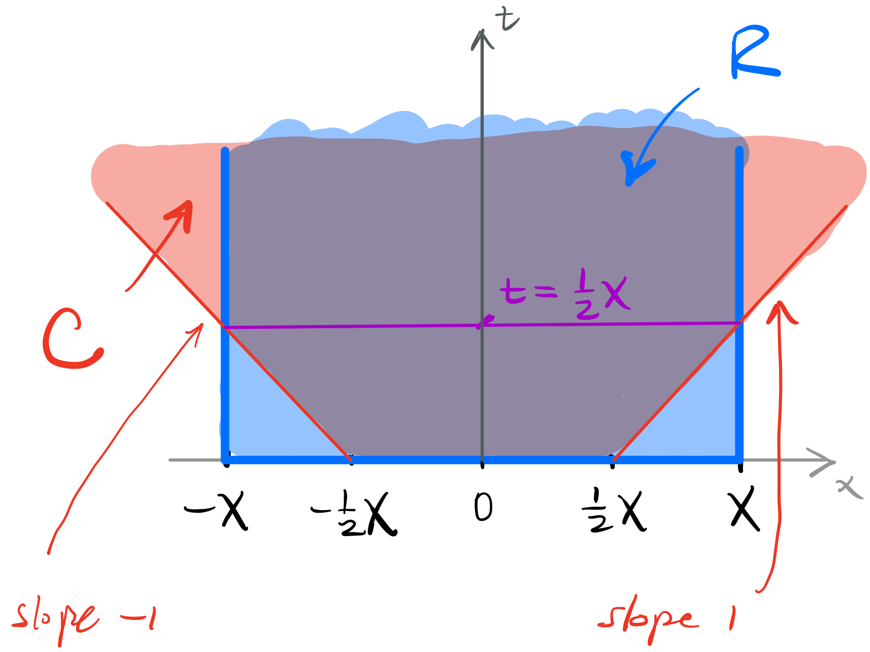
the domain of influence C intersects Γ.

- Look at the small rectangle $[x_0, x_1] \times [0, t_1]$: for this rectangle $u = 0$ on the bottom, right, left, but u need not be 0 in C.

<u>Detailed proof:</u> For convenience, assume that $c = 1$.

In the sketch above take $x_1 = X > 0$ and take $x_0 = -X$. Take $y_1 = \frac{1}{2}X$, $y_0 = -\frac{1}{2}X$.

Take $t_1 \in (\frac{1}{4}X, \frac{1}{2}X)$.



Take $\phi(x) = 0 \quad \forall x$

Take $\psi(x) = \begin{cases} 1 & |x| < \frac{1}{2}X \\ 0 & |x| \geq \frac{1}{2}X \end{cases}$

Then u must be 0 outside of C.

Consider the rectangle

$$[-X, X] \times [0, t_1]$$

Then u is 0 on its bottom, right, left.

However, at $(x,t) = (0, \frac{1}{4}X)$ we have:

$$u(0, \tfrac{1}{4}X) = \int_{-\frac{1}{4}X}^{\frac{1}{4}X} ds = \frac{1}{2}X > 0.$$

This means that there is no max principle.

Section 1.5 Q1: Consider the problem

$$\begin{cases} u''(x) + u(x) = 0 & 0<x<L \\ u(0) = u(L) = 0 \end{cases}$$

$u(x) \equiv 0$ is a solution. Is this solution unique or not? Does the answer depend on L?

We know that the equation $u'' = -u$ is satisfied by sines and cosines. By linearity, we can take linear combinations, so a general solution will have the form: $u(x) = A\cos x + B\sin x$.

Now we need to satisfy the boundary conditions.

$$0 = u(0) = A \underbrace{\cos 0}_{=1} + B \underbrace{\sin 0}_{=0} = A \Rightarrow A = 0$$

So we discover that $A=0$.

$$0 = u(L) = B\sin L \Longrightarrow \left\{ \begin{matrix} B=0 \\ \text{OR} \\ \sin L = 0 \end{matrix} \right\}$$

$B=0$ means that $u \equiv 0$. So we focus on $\sin L = 0$. This can only hold if L is a multiple of π.

Conclusion: If $L = n\pi$ then any function of the form $B\sin x$ is a solution, so solutions aren't unique.

If L is not a multiple of π then $u \equiv 0$ is the only solution.

Section 1.5 Q2: Consider the problem

$$\begin{cases} u''(x) + u'(x) = f(x) & 0<x<l \\ u'(0) = u(0) = \frac{1}{2}\left(u'(l)+u(l)\right) \end{cases}$$

with $f(x)$ a given function.

(a) Is the solution unique? Explain.

(b) Does a solution necessarily exist, or is there a condition that $f(x)$ must satisfy for existence.

(a) Suppose that there exist two solutions u and v. Noticing that the equation is linear, $w = u - v$ will be a solution of

$$\begin{cases} w''(x) + w'(x) = 0 & 0<x<l \\ w'(0) = w(0) = \frac{1}{2}\left(w'(l)+w'(0)\right) \end{cases}$$

Solutions of the original problem are unique if and only if $w \equiv 0$ is the only solution of this problem.

Integrating once we find $w'(x) + w(x) = C$

$$\Rightarrow \quad \frac{dw}{dx} = -w + C$$

$$\Rightarrow \quad \frac{dw}{-w+C} = dx$$

integrate $\Rightarrow \quad -\log(-w+C) = x + D$

$$\Longrightarrow \quad \log(-w+C) = -x-D$$

exponentiate $\longrightarrow \quad e^{\log(-w+C)} = e^{-x-D} = e^{-x}e^{-D}$

Since $\quad e^{\log(-w+C)} = -w+C \quad$ we find that

$$-w+C = e^{-x}e^{-D}$$

$$\Longrightarrow \quad w = C - e^{-x}e^{-D}$$

Denoting $\quad c_1 = C \qquad c_2 = -e^{-D} \quad$ we have:

$$w(x) = c_1 + c_2 e^{-x} \qquad \text{and} \quad w'(x) = -c_2 e^{-x}$$

Now for the boundary conditions:

We compute: $\qquad w(0) = c_1 + c_2 \qquad w(\ell) = c_1 + c_2 e^{-\ell}$

$\qquad\qquad\qquad\qquad w'(0) = -c_2 \qquad w'(\ell) = -c_2 e^{-\ell}$

We need to satisfy $\quad w'(0) \overset{(1)}{=} w(0) \overset{(2)}{=} \frac{1}{2}(w'(\ell)+w(\ell))$.

(1) $\Longrightarrow \quad c_1 + c_2 = -c_2 \quad \Longrightarrow \quad c_1 = -2c_2$

Plugging this into $\frac{1}{2}(w'(\ell)+w(\ell))$ we have:

$$\frac{1}{2}\left(-2c_2 + c_2 e^{-\ell} - c_2 e^{-\ell}\right) = -c_2$$

which is indeed $w(0)$, so (2) holds!

Conclusion: $w(x) = -2C_2 + C_2 e^{-x}$

solves the eq. for $w = u - v$ with the required boundary conditions. Hence, the difference between u and v isn't necessarily 0, so that they can be different from one another.

$\longrightarrow$ The solution of the original problem isn't unique.

(b) Now we return to $u''(x) + u'(x) = f(x) \quad 0 < x < l$ and ask whether a solution necessarily exists. There isn't much to do with this eq. other than to integrate it from 0 to l:

$$\int_0^l (u''(x) + u'(x))\, dx = \int_0^l f(x)\, dx$$

$$u'(x)\Big|_{x=0}^{l} + u(x)\Big|_{x=0}^{l} = \int_0^l f(x)\, dx$$

$$u'(l) + u(l) - (u'(0) + u(0)) = \int_0^l f(x)\, dx \quad (\ast)$$

The given boundary condition $u'(0) = u(0) = \frac{1}{2}(u'(l) + u(l))$ implies that the LHS of $(\ast)$ is 0.

We find that $\int_0^l f(x)\, dx = 0$ is a necessary condition for a solution to exist.

Section 1.5 Q5: Consider the equation

$$\begin{cases} u_x + y u_y = 0 \\ u(x,0) = \phi(x) \end{cases}$$

(a) For $\phi(x) \equiv x$ show that no solution exists.

(b) For $\phi(x) \equiv 1$ show that there are many solutions.

This equation appears in Section 1.2 of the book, but let's solve here too. We know that for a first-order PDE with variable coefficients we want want to solve

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

In this case, it is: $\frac{dy}{dx} = \frac{y}{1} \implies \frac{dy}{y} = dx$

integrate $\implies$ $\log y = x + C$

exponentiate $\implies$ $\underbrace{e^{\log y}}_{=y} = e^{x+C} = e^x \underbrace{e^C}$ Denote this const A

So we find that $y(x) = Ae^x$ solves this, and gives an eq for the characteristic curves.

We exrese the constat as: $A = ye^{-x}$

so that

$$u(x,y) = f(ye^{-x})$$

is the general form of the solution of the PDE.

(a) Let's see what happens with $\phi(x) = x$:

$$x = \phi(x) = u(x,0) = f(0)$$

This must hold for every x, which is impossible since $f(0)$ is some fixed number.
So there are no solutions with $\phi(x) \equiv x$.

(b) Try $\phi(x) \equiv 1$:

$$1 = \phi(x) = u(x,0) = f(0)$$

So any $u(x,y)$ of the form $u(x,y) = f(ye^{-x})$ satisfying $f(0) = 1$ will solve the equation.
$\Longrightarrow$ There are many functions f that satisfy this.