Section 2.1 Q1: Solve $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$

D'Alembert's formula tells us that

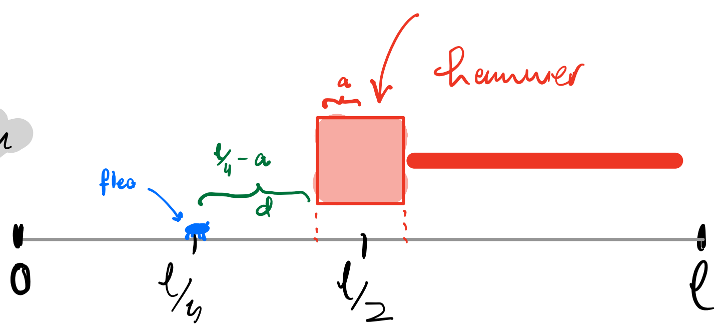
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

$$= \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\, ds$$

$$= \frac{1}{2} e^x \left(e^{ct} + e^{-ct}\right) - \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

$$= e^x \cosh(ct) + \frac{1}{c}\sin(ct)\sin x$$

Section 2.1 Q3: The midpoint of a piano string of tension T, density ρ, and length ℓ is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $\ell/4$ from one end ($a < \ell/4$). How long does it take the disturbance to reach the flea?



The flea is distanced $d = \frac{\ell}{4} - a$ from where the hammer hits. The disturbance travels at $c = \sqrt{\frac{T}{\rho}}$. The time it would take is:

$$t = \frac{d}{c} = \sqrt{\frac{\rho}{T}}\left(\frac{\ell}{4} - a\right)$$

Section 2.1 Q7: If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Hence:

$$\begin{aligned} u(-x,t) &= \frac{1}{2}\left[\phi(-x+ct) + \phi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds \\ &= \frac{1}{2}\left[-\phi(x-ct) - \phi(x+ct)\right] - \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)\,ds \\ &= -\frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \\ &= -u(x,t). \end{aligned}$$

Let's explain what we did with the integral of ψ:

$$\frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds = -\frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)\,ds$$

here we just used the oddness of ψ: $\psi(s) = -\psi(-s)$.

$$-\frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)\,ds = -\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

here we use the fact that the integration interval $[-x-ct, -x+ct]$ becomes $[x+ct, x-ct]$ by changing s to $-s$, and then $\psi(-s)$ becomes $-\psi(s)$. We get another $-$ in front by changing $[x+ct, x-ct]$ to $[x-ct, x+ct]$.

Section 2.2 Q1: Use the energy conservation of the wave eq. to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$.

We know that $E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx$ is conserved.

At time $t=0$ we have:

$$E = \frac{1}{2}\int_{-\infty}^{\infty} [\rho \underbrace{\psi^2(x)}_{0} + T \underbrace{\phi_x^2(x)}_{0}]\, dx = 0.$$

So for any later time we have

$$0 = E = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{(\rho u_t^2 + T u_x^2)}_{\geq 0}\, dx$$

How can the integral of a non-negative function be 0?

By the first vanishing theorem, the only way for this to happen is that the integrand is, in fact, 0:

$$\rho u_t^2 + T u_x^2 = 0 \qquad \forall x.$$

But since both terms are non-negative, this means that they are both 0 individually: $\rho u_t^2 = T u_x^2 = 0$.

Since $\rho > 0$, $T > 0$ we have $u_t^2 = u_x^2 = 0 \implies u_t = u_x = 0$

Hence u must be constant. Since it is 0 initially, it is 0 everywhere.

Section 2.2 Q3: Show that the wave eq. has the following invariance properties:

a) Any translate $u(x-y,t)$, y fixed, is a solution.

b) Any derivative of a solution is a solution.

c) The dilated function $u(ax,at)$ is also a solution $\forall a$.

We start with $u(x,t)$ which is a solution: $u_{tt} = c^2 u_{xx}$.

a) Define $w(x,t) = u(x-y,t)$. Then:

$$w_{tt}(x,t) = u_{tt}(x-y,t)$$

$$w_{xx}(x,t) = u_{xx}(x-y,t)$$

$$\Rightarrow \quad w_{tt} - c^2 w_{xx} = u_{tt} - c^2 u_{xx} = 0.$$

b) Define $w(x,t) = \partial_x u(x,t)$.

$$w_{tt}(x,t) = \partial_{tt}(\partial_x u(x,t)) = \partial_x (u_{tt}(x,t))$$

$$w_{xx}(x,t) = \partial_{xx}(\partial_x u(x,t)) = \partial_x (u_{xx}(x,t))$$

$$\Rightarrow \quad w_{tt} - c^2 w_{xx} = \partial_x [\underbrace{u_{tt} - c^2 u_{xx}}_{=0}] = 0$$

Since any other derivative will commute with ∂_{xx} and ∂_{tt}, the same result will hold for any other derivative (or combination of derivatives) in x, t.

c) Define $w(x,t) = u(ax,at)$. Then

$$w_x = a u_x, \quad w_{xx} = a^2 u_x, \quad w_t = a u_t, \quad w_{tt} = a^2 u_{tt}$$

$$\Rightarrow \quad w_{tt} - c^2 w_{xx} = a^2 [\underbrace{u_{tt} - c^2 u_{xx}}_{0}] = 0.$$

Section 2.2 Q5: For the damped string satisfying

$$u_{tt} - c^2 u_{xx} + r u_t = 0 \qquad (r > 0)$$

show that the energy decreases.

Recall that the energy is defined as:

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx .$$

Then: $$E'(t) = \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{xt})\,dx$$

plugging in the eq.

$$= \int_{-\infty}^{\infty} (\rho u_t \{c^2 u_{xx} - r u_t\} + T u_x u_{xt})\,dx$$

$$= \int_{-\infty}^{\infty} (\rho \tfrac{T}{\rho} u_t u_{xx} - r\rho u_t^2 + T u_x u_{xt})\,dx$$

integration by parts

$$= -\int_{-\infty}^{\infty} T \cancel{u_{tx} u_x}\,dx + \underbrace{\cancel{[T u_t u_x]_{x=-\infty}^{\infty}}} - \int_{-\infty}^{\infty} r\rho u_t^2\,dx + \int_{-\infty}^{\infty} T \cancel{u_x u_{xt}}\,dx$$

This vanishes since u and its derivatives are assumed to vanish "at" $\pm\infty$

$$= -\int_{-\infty}^{\infty} \underbrace{r\rho u_t^2}\,dx \le 0$$

the integrand is non-negative

$$E'(t) \le 0 \implies \text{the energy decreases!}$$